Part II

Euclidean vector spaces.

1. Introduction to Euclidean spaces.

1 Scalar product.

1.1 Definition.

Definition 1.1 *Let U be a real vector space. A bilinear symmetric and positive definite form is called a* **scalar product** *or* **dot product**.

If $f : U \times U \longrightarrow \mathbb{R}$ is a bilinear form corresponding to a scalar product, the image by f of a pair of vectors $(\bar{x}, \bar{y})$ will normally be denoted as

$$f(\bar{x}, \bar{y}) = \bar{x} \cdot \bar{y}.$$

With this notation, the properties that must be fulfilled for f to effectively define a scalar product can be written as follows:

1. **Bilinearity:** For any $\bar{x}, \bar{y}, \bar{z} \in U$, $\alpha, \beta \in \mathbb{R}$, the following must hold:
$$(\alpha\bar{x} + \beta\bar{y}) \cdot \bar{z} = \alpha\bar{x} \cdot \bar{z} + \beta\bar{y} \cdot \bar{z}.$$
$$\bar{x} \cdot (\alpha\bar{y} + \beta\bar{z}) = \alpha\bar{x} \cdot \bar{y} + \beta\bar{x} \cdot \bar{z}.$$
2. **Symmetry:** For any $\bar{x}, \bar{y} \in U$, we must have
$$\bar{x} \cdot \bar{y} = \bar{y} \cdot \bar{x}.$$
3. **Positive definiteness:** For any $\bar{x} \in U$ different from $\bar{0}$
$$\bar{x} \cdot \bar{x} > 0.$$

Definition 1.2 *An* **Euclidean vector space** *(or just* **Euclidean space***) is a real vector space on which we have defined a scalar product.*

1.2 Gram matrix of a scalar product.

Definition 1.3 *Let U be an Euclidean vector space. Let $B = \{\bar{u}_1, \dots, \bar{u}_n\}$ be a basis of U. The matrix G_B associated to the bilinear form f with respect to the basis B is called the* **Gram matrix** *of the scalar product with respect to this basis.*

With the notation we have adopted for the scalar product, the Gram matrix with respect to the basis B is of the form:

$$G_B = \begin{pmatrix} \bar{u}_1 \cdot \bar{u}_1 & \bar{u}_1 \cdot \bar{u}_2 & \dots & \bar{u}_1 \cdot \bar{u}_n \\ \bar{u}_2 \cdot \bar{u}_1 & \bar{u}_2 \cdot \bar{u}_2 & \dots & \bar{u}_2 \cdot \bar{u}_n \\ \vdots & \vdots & \ddots & \vdots \\ \bar{u}_n \cdot \bar{u}_1 & \bar{u}_n \cdot \bar{u}_2 & \dots & \bar{u}_n \cdot \bar{u}_n \end{pmatrix}.$$

The matrix expression of the scalar product with respect to this basis is:

$$\boxed{\bar{x} \cdot \bar{y} = (x)^t G(y)}$$

where $(x^i), (y^i)$ are the coordinates of the vectors $\bar{x}, \bar{y}$ with respect to the basis B.

Taking into account that a scalar product corresponds to a symmetric and positive definite bilinear form, the Gram matrix fulfills the following properties:

1. *G_B is symmetric.*
2. *G_B is positive definite.*
3. *G_B is congruent with the identity. That is, there always exists a basis with respect to which the Gram matrix of the scalar product is the identity.*
   Proof: Recall that as we have seen, every symmetric and positive definite matrix is congruent to the identity.
4. *If B and B' are two bases of the vector space U, then:*
$$\boxed{G'_B = (M_{BB'})^t G_B M_{BB'}}$$
5. *The Gram matrix is a fully covariant, second-order homogeneous tensor.*
   Proof: Just take into account the way it behaves under a change of basis.

2 Norm of a vector.

Definition 2.1 *Let U be a Euclidean vector space. The* **norm** *or* **modulus** *of a vector $\bar{x}$ is defined as*
$$\|\bar{x}\| = +\sqrt{\bar{x} \cdot \bar{x}}.$$

Since the scalar product is a positive definite symmetric bilinear form, $\bar{x} \cdot \bar{x}$ is always nonnegative, so this square root is a real number.

Let's see some fundamental properties of the norm:

1. $\|\bar{x}\| = 0 \iff \bar{x} = \bar{0}$.
   Proof:
$$\|\bar{x}\| = 0 \iff \bar{x} \cdot \bar{x} = 0 \underset{\uparrow}{\iff} \bar{x} = \bar{0}$$
since the scalar product is positive definite

2. $\|\bar{x}\| > 0$ *for any* $\bar{x} \neq \bar{0}$.
3. $\|\lambda\bar{x}\| = |\lambda|\|\bar{x}\|$, *for any* $\lambda \in \mathbb{R}$, $\bar{x} \in U$. **Proof:**
$$\|\lambda\bar{x}\| = +\sqrt{(\lambda\bar{x}) \cdot (\lambda\bar{x})} = +\sqrt{\lambda^2\bar{x} \cdot \bar{x}} = +|\lambda|\sqrt{\bar{x} \cdot \bar{x}} = |\lambda|\|\bar{x}\|$$
4. *Schwarz Inequality:*
$$\|\bar{x}\|\|\bar{y}\| \geq |\bar{x} \cdot \bar{y}|$$
for any $\bar{x}, \bar{y} \in U$.
Proof I: We distinguish two possibilities:
- If $\bar{x}, \bar{y}$ are dependent, either $\bar{x} = \bar{0}$ (and the result is immediate), or $\bar{y} = \lambda\bar{x}$, for some $\lambda \in \mathbb{R}$. Then:
$$\|\bar{x}\|\|\bar{y}\| = \|\bar{x}\|\|\lambda\bar{x}\| = |\lambda|\|\bar{x}\|^2$$
and
$$|\bar{x} \cdot \bar{y}| = |\bar{x} \cdot (\lambda\bar{x})| = |\lambda||\bar{x} \cdot \bar{x}| = |\lambda|\|\bar{x}\|^2.$$
- If they are independent, we can consider the restriction of the scalar product to the subspace $V = \mathcal{L}\{\bar{x}, \bar{y}\}$ generated by both. The restriction of the scalar product to V is still a positive definite bilinear form. The matrix of this bilinear form with respect to the basis $\{\bar{x}, \bar{y}\}$ is:
$$G' = \begin{pmatrix} \bar{x} \cdot \bar{x} & \bar{x} \cdot \bar{y} \\ \bar{y} \cdot \bar{x} & \bar{y} \cdot \bar{y} \end{pmatrix} = \begin{pmatrix} \|\bar{x}\|^2 & \bar{x} \cdot \bar{y} \\ \bar{x} \cdot \bar{y} & \|\bar{y}\|^2 \end{pmatrix}$$
Since it is positive definite, its determinant is a positive number:
$$|G'| > 0 \quad \Rightarrow \quad \|\bar{x}\|^2\|\bar{y}\|^2 > (\bar{x} \cdot \bar{y})^2 \quad \Rightarrow \quad \|\bar{x}\|\|\bar{y}\| > |\bar{x} \cdot \bar{y}|.$$
Proof II: We consider the vector $\bar{x}+\lambda\bar{y}$ for any $\lambda \in \mathbb{R}$. Since the scalar product is definite positive we have
$$\begin{aligned}(\bar{x} + \lambda\bar{y}) \cdot (\bar{x} + \lambda\bar{y}) > 0 \quad &\Rightarrow \quad \bar{x} \cdot \bar{x} + \lambda\bar{x} \cdot \bar{y} + \lambda\bar{y} \cdot \bar{x} + \lambda^2\bar{y} \cdot \bar{y} \geq 0 \quad \Rightarrow \\ &\Rightarrow \quad \|\bar{y}\|^2\lambda^2 + 2(\bar{x} \cdot \bar{y})\lambda + \|\bar{x}\|^2 \geq 0.\end{aligned}$$
That is, we obtain a quadratic equation in λ that never takes negative values. Therefore it can have at most one real (double) solution and the discriminant of that equation cannot be positive:
$$(2(\bar{x} \cdot \bar{y}))^2 - 4\|\bar{y}\|^2\|\bar{x}\|^2 \leq 0 \quad \Rightarrow \quad \|\bar{x}\|\|\bar{y}\| \geq |\bar{x} \cdot \bar{y}|.$$
5. *Minkowski inequality:*
$$\|\bar{x} + \bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\|$$
for any $\bar{x}, \bar{y} \in U$.
Proof: We have:
$$\begin{aligned}\|\bar{x} + \bar{y}\|^2 &= (\bar{x} + \bar{y}) \cdot (\bar{x} + \bar{y}) = \bar{x} \cdot \bar{x} + \bar{x} \cdot \bar{y} + \bar{y} \cdot \bar{x} + \bar{y} \cdot \bar{y} = \\ &= \|\bar{x}\|^2 + \|\bar{y}\|^2 + 2\bar{x} \cdot \bar{y} \leq \|\bar{x}\|^2 + \|\bar{y}\|^2 + 2|\bar{x} \cdot \bar{y}|\end{aligned}$$
and applying Schwarz inequality
$$\|\bar{x} + \bar{y}\|^2 \leq \|\bar{x}\|^2 + \|\bar{y}\|^2 + 2\|\bar{x}\|\|\bar{y}\| \leq (\|\bar{x}\| + \|\bar{y}\|)^2.$$

Definition 2.2 *In an Euclidean space* U*, a vector of norm* 1 *is called a* **unit vector** *or* **unitary vector.**

3 Angle between two vectors.

Definition 3.1 *Let* U *be a Euclidean space. Given two non-zero vectors* $\bar{x}, \bar{y} \in U$ *we define the* **angle** *formed by them as the real number* $(\bar{x}, \bar{y}) \in [0, \pi]$ *satisfying:*
$$cos(\bar{x}, \bar{y}) = \frac{\bar{x} \cdot \bar{y}}{\|\bar{x}\|\|\bar{y}\|}.$$

By the Schwarz inequality:
$$-1 \leq \frac{\bar{x} \cdot \bar{y}}{\|\bar{x}\|\|\bar{y}\|} \leq 1.$$
Therefore, there is only one angle in the interval $[0, \pi]$ whose cosine is the previous one.

This definition of angle implies the following indentity:
$$\boxed{\bar{x} \cdot \bar{y} = \|\bar{x}\| \, \|\bar{y}\| \, cos(\bar{x}, \bar{y})}$$