Part I

Bilinear mappings and homogeneous tensors.

1. Bilinear mappings and quadratic forms.

1 Bilinear mappings.

Definition 1.1 *Given three vector spaces U, V, W over the field $\mathbb{K}$, we say that the mapping*

$$f : U \times V \longrightarrow W$$

is **bilinear** *if it is linear on each one of its arguments, i. e.*

$$f(\alpha\bar{u}_1 + \beta\bar{u}_2, \bar{v}_1) = \alpha f(\bar{u}_1, \bar{v}_1) + \beta f(\bar{u}_2, \bar{v}_1)$$
$$f(\bar{u}_1, \alpha\bar{v}_1 + \beta\bar{v}_2) = \alpha f(\bar{u}_1, \bar{v}_1) + \beta f(\bar{u}_1, \bar{v}_2)$$

for any $\alpha, \beta \in \mathbb{K}$, $\bar{u}_1, \bar{u}_2 \in U$, $\bar{v}_1, \bar{v}_2 \in V$.

Definition 1.2 *A* **bilinear form** *is a bilinear mapping whose final vector space is the field $\mathbb{K}$:*

$$f : U \times V \longrightarrow \mathbb{K}, \text{ bilinear.}$$

1.1 The matrix of a bilinear form.

Assume that U, V are vector spaces, B_1 is a basis of U and B_2 a basis of V.

$$B_1 = \{\bar{u}_1, \dots, \bar{u}_m\}, \qquad B_2 = \{\bar{v}_1, \dots, \bar{v}_n\}.$$

Let $f : U \times V \longrightarrow \mathbb{K}$ be a bilinear form. Fix vectors $\bar{x} \in U$, $\bar{y} \in V$ with the following expression in coordinates relative to the bases B_1, B_2 :

$$\bar{x} = x^i\bar{u}_i; \qquad \bar{y} = y^j\bar{v}_j.$$

Since f is bilinear we have

$$f(\bar{x}, \bar{y}) = f(x^i\bar{u}_i, y^j\bar{v}_j) = x^i y^j f(\bar{u}_i, \bar{v}_j)$$

The matrix associated to f relative to the bases B_1 and B_2 is defined as

$$F_{B_1B_2} = \begin{pmatrix} f(\bar{u}_1, \bar{v}_1) & f(\bar{u}_1, \bar{v}_2) & \dots & f(\bar{u}_1, \bar{v}_n) \\ f(\bar{u}_2, \bar{v}_1) & f(\bar{u}_2, \bar{v}_2) & \dots & f(\bar{u}_2, \bar{v}_n) \\ \vdots & \vdots & \ddots & \vdots \\ f(\bar{u}_m, \bar{v}_1) & f(\bar{u}_m, \bar{v}_2) & \dots & f(\bar{u}_m, \bar{v}_n) \end{pmatrix},$$

Hence we can write

$$f(\bar{x}, \bar{y}) = (\, x^1 \quad x^2 \quad \dots \quad x^m \,) \begin{pmatrix} f(\bar{u}_1, \bar{v}_1) & f(\bar{u}_1, \bar{v}_2) & \dots & f(\bar{u}_1, \bar{v}_n) \\ f(\bar{u}_2, \bar{v}_1) & f(\bar{u}_2, \bar{v}_2) & \dots & f(\bar{u}_2, \bar{v}_n) \\ \vdots & \vdots & \ddots & \vdots \\ f(\bar{u}_m, \bar{v}_1) & f(\bar{u}_m, \bar{v}_2) & \dots & f(\bar{u}_m, \bar{v}_n) \end{pmatrix} \begin{pmatrix} y^1 \\ y^n \\ \vdots \\ y^n \end{pmatrix}$$

or just

$$f(\bar{x}, \bar{y}) = (x)^t F_{B_1B_2}(y).$$

1.2 Change of basis of the matrix associated to a bilinear form.

Let U, V be vector spaces. We fix the bases

$$\left.\begin{array}{l} B_1 = \{\bar{u}_1, \dots, \bar{u}_m\} \\ B_1' = \{\bar{u}_1', \dots, \bar{u}_m'\} \end{array}\right\} \text{Bases of } U; \qquad \left.\begin{array}{l} B_2 = \{\bar{v}_1, \dots, \bar{v}_n\} \\ B_2' = \{\bar{v}_1', \dots, \bar{v}_n'\} \end{array}\right\} \text{Bases of } V.$$

We will denote the coordinates of the vectors $\bar{x} \in U$, $\bar{y} \in V$ relative to each one of these bases as follows:

Coordinates of $\bar{x}$.		Coordinates of $\bar{y}$.	
$(x^1, \dots, x^m)$	relative to the basis B_1	$(y^1, \dots, y^n)$	relative to the basis B_2
$(x'^1, \dots, x'^m)$	relative to the basis B_1'	$(y'^1, \dots, y'^n)$	relative to the basis B_2'

The relations between the different bases and coordinates, in terms of the corresponding change-of-basis matrices are as follows:

$$\begin{array}{l|l} (\bar{u}') = (\bar{u})M_{B_1B_1'} & (\bar{v}') = (\bar{v})M_{B_2B_2'} \\ (x) = M_{B_1B_1'}(x') & (y) = M_{B_2B_2'}(y') \end{array}$$

Moreover, we know how to express the bilinear mapping f by means of a matrix, either in terms of the bases B_1, B_2 or the bases B_1', B_2':

$$f(\bar{x}, \bar{y}) = (x)^t F_{B_1B_2}(y) \;\Big|\; f(\bar{x}, \bar{y}) = (x')^t F_{B_1'B_2'}(y')$$

Let us see how the matrices associated to f relative to the bases B_1, B_2 and B_1', B_2' are related to each other:

$$(x)^t F_{B_1B_2}(y) = (M_{B_1B_1'}(x'))^t F_{B_1B_2} M_{B_2B_2'}(y') = (x')^t (M_{B_1B_1'})^t F_{B_1B_2} M_{B_2B_2'}(y')$$

We deduce

$$\boxed{F_{B_1'B_2'} = (M_{B_1B_1'})^t F_{B_1B_2} M_{B_2B_2'}}$$

2 Bilinear forms on a single vector space.

We are especially interested in bilinear forms that are defined on a single vector space, i. e. bilinear mappings of the form

$$f : U \times U \longrightarrow \mathbb{K}.$$

Let us analyze their main features.

2.1 Matrices and change of basis.

In this case, in order to find the matrix of a bilinear form $f : U \times U \longrightarrow \mathbb{K}$ it is enough to fix a single basis

$$B = \{\bar{u}_1, \dots, \bar{u}_n\}$$

of U. We will have

$$f(\bar{x}, \bar{y}) = (x)^t F_{BB}(y), \text{ where } F_{BB} = \begin{pmatrix} f(\bar{u}_1, \bar{u}_1) & f(\bar{u}_1, \bar{u}_2) & \dots & f(\bar{u}_1, \bar{u}_n) \\ f(\bar{u}_2, \bar{u}_1) & f(\bar{u}_2, \bar{u}_2) & \dots & f(\bar{u}_2, \bar{u}_n) \\ \vdots & \vdots & \ddots & \vdots \\ f(\bar{u}_n, \bar{u}_1) & f(\bar{u}_n, \bar{u}_2) & \dots & f(\bar{u}_n, \bar{u}_n) \end{pmatrix}$$

In this case the matrix F_{BB} can be also denoted by F_B or simply by F, if we specify in advance which basis B we are using.

If we fix another basis of the vector space U:

$$B' = \{\bar{u}'_1, \dots, \bar{u}'_n\},$$

we can write the change of basis as

$$\boxed{F_{B'} = (M_{BB'})^t F_B M_{BB'}}$$

Consequently we deduce:

Two matrices associated to the same bilinear form on a vector space U are congruent.

2.2 The vector space of all bilinear forms on U.

We denote by $Bil(U)$ the set of all bilinear forms of the form $f : U \times U \to \mathbb{K}$. It is easy to see that $Bil(U)$ is a vector space with the operations of sum and multiplication by scalars defined as usual.

Let us find its dimension. To this end we will define an isomorphism between $Bil(U)$ and $M_{n\times n}(\mathbb{K})$. Once fixed a basis B of U we define

$$\begin{array}{rccc} \pi : & Bil(U) & \longrightarrow & M_{n\times n}(\mathbb{K}) \\ & f & \longrightarrow & F_{BB} \end{array}$$

This mapping satisfies the following properties:

- It is linear, since if $f, g \in Bil(U)$, $\alpha, \beta \in \mathbb{K}$ and $\bar{x}, \bar{y} \in U$ we have

$$(\alpha f + \beta g)(\bar{x}, \bar{y}) = \alpha f(\bar{x}, \bar{y}) + \beta g(\bar{x}, \bar{y}) = \alpha (x)^t F(y) + \beta (x)^t G(y) = (x)^t (\alpha F + \beta G)(y)$$

and hence $\pi(\alpha f + \beta g) = \alpha\pi(f) + \beta\pi(g)$.

- It is injective, since if two bilinear forms are associated to the same matrix, then they actually coincide as mappings.

- It is onto, since given any $n \times n$ matrix A we can always define a bilinear form whose matrix relative to the basis B is exactly A:

$$f(\bar{x}, \bar{y}) = (x)^t A(y)$$

Hence $dim(Bil(V)) = dim(M_{n\times n}(\mathbb{K})) = n^2$.

2.3 Symmetric and antisymmetric bilinear forms

Definition 2.1 *Let $f : U \times U \longrightarrow \mathbb{K}$ be a bilinear form. We say that*

- f is **symmetric** *if $f(\bar{x}, \bar{y}) = f(\bar{y}, \bar{x})$ for any $\bar{x}, \bar{y} \in U$.*

- f is **antisymmetric** *if $f(\bar{x}, \bar{y}) = -f(\bar{y}, \bar{x})$ for any $\bar{x}, \bar{y} \in U$.*

We will denote by $Bil_S(U)$ the set of all symmetric bilinear forms on U.

We will denote by $Bil_A(U)$ the set of all antisymmetric bilinear forms on U.

Let us see some properties of these types of bilinear forms:

1. *$Bil_S(U)$ is a vector subspace of $Bil(U)$.*

   Proof: To begin with, $Bil_S(U) \neq \emptyset$ since the null bilinear form is symmetric. Moreover, if $f, g \in Bil_S(U)$, $\alpha, \beta \in \mathbb{K}$ and $\bar{x}, \bar{y} \in U$ we have

   $$\begin{aligned}(\alpha f + \beta g)(\bar{x}, \bar{y}) &= \alpha f(\bar{x}, \bar{y}) + \beta g(\bar{x}, \bar{y}) = \alpha f(\bar{y}, \bar{x}) + \beta g(\bar{y}, \bar{x}) = \\ &= (\alpha f + \beta g)(\bar{y}, \bar{x})\end{aligned}$$

   and hence $(\alpha f + \beta g) \in Bil_S(U)$.

2. *$Bil_S(U)$ is a vector subspace of $Bil(U)$.*

   Proof: As above, $Bil_A(U) \neq \emptyset$, since the bilinear form 0 is antisymmetric. Moreover if $f, g \in Bil_A(U)$, $\alpha, \beta \in \mathbb{K}$ and $\bar{x}, \bar{y} \in U$ we have

   $$\begin{aligned}(\alpha f + \beta g)(\bar{x}, \bar{y}) &= \alpha f(\bar{x}, \bar{y}) + \beta g(\bar{x}, \bar{y}) = -\alpha f(\bar{y}, \bar{x}) - \beta g(\bar{y}, \bar{x}) = \\ &= -(\alpha f + \beta g)(\bar{y}, \bar{x})\end{aligned}$$

   and hence $(\alpha f + \beta g) \in Bil_A(U)$.

3. *The subspaces $Bil_S(U)$ and $Bil_A(U)$ are complementary in $Bil(U)$.*

   Proof: We have
   - $Bil_S(U) \cap Bil_A(U) = \{0\}$ since if a bilinear form f is simultaneously symmetric and antisymmetric, then it satisfies

   $$f(\bar{x}, \bar{y}) = f(\bar{y}, \bar{x}) = -f(\bar{x}, \bar{y}) \quad \Rightarrow \quad f(\bar{x}, \bar{y}) = 0$$

   for any $\bar{x}, \bar{y} \in U$.
   - Any bilinear form f can be decomposed into a sum of a symmetric bilinear form f_S and an antisymmetric one f_A, defined as follows:

   $$f_S(\bar{x}, \bar{y}) = \frac{1}{2}(f(\bar{x}, \bar{y}) + f(\bar{y}, \bar{x}))$$
   $$f_A(\bar{x}, \bar{y}) = \frac{1}{2}(f(\bar{x}, \bar{y}) - f(\bar{y}, \bar{x}))$$

4. *If f is a symmetric bilinear form, the matrix F associated to f with respect to any basis of U is symmetric.*

   Proof: Fix a basis $B = \{\bar{u}_1, \ldots, \bar{u}_n\}$. For any i, j with $1 \leq i, j \leq n$ we have

   $$f_{ij} = f(\bar{u}_i, \bar{u}_j) = f(\bar{u}_j, \bar{u}_i) = f_{ji}.$$

5. *If f is an antisymmetric bilinear form, the matrix F associated to f with respect to any basis of U is antisymmetric.*

   Proof: Fix a basis $B = \{\bar{u}_1, \ldots, \bar{u}_n\}$. For any i, j with $1 \leq i, j \leq n$ we have

   $$f_{ij} = f(\bar{u}_i, \bar{u}_j) = -f(\bar{u}_j, \bar{u}_i) = -f_{ji}.$$

3 Quadratic forms.

3.1 Definition.

Definition 3.1 *We will give two equivalent definitions:*

1. *Given any vector space U and any symmetric bilinear form $f : U \times U \longrightarrow \mathbb{K}$, we call the* **quadratic form associated to** *f the mapping*

   $$\begin{array}{rccc} \omega : & U & \longrightarrow & \mathbb{K} \\ & \bar{x} & \longrightarrow & f(\bar{x}, \bar{x}) \end{array}$$

2. *Given any vector space U a mapping $\omega : U \longrightarrow \mathbb{K}$ is a* **quadratic form** *if it satisfies the following conditions:*
   - *$\omega(\lambda\bar{x}) = \lambda^2\omega(\bar{x})$, for any $\lambda \in \mathbb{K}$, $\bar{x} \in U$.*
   - *the mapping $g : U \times U \longrightarrow K$ defined as*

   $$g(\bar{x}, \bar{y}) = \frac{1}{2}(\omega(\bar{x} + \bar{y}) - \omega(\bar{x}) - \omega(\bar{y}))$$

   is bilinear and symmetric. (This mapping is called the **polar form** *of ω.)*

Let us see that these definitions are indeed equivalent:

1) $\Rightarrow$ 2). If $f : U \times U \longrightarrow \mathbb{K}$ is a symmetric bilinear form and ω its associated quadratic form, we have

- $\omega(\lambda\bar{x}) = f(\lambda\bar{x}, \lambda\bar{x}) = \lambda^2 f(\bar{x}, \bar{x}) = \lambda^2\omega(\bar{x})$.
- If $\omega(\bar{x}) = f(\bar{x}, \bar{x})$ and g is defined as above,

$$\begin{aligned} g(\bar{x}, \bar{y}) &= \frac{1}{2}(\omega(\bar{x} + \bar{y}) - \omega(\bar{x}) - \omega(\bar{y})) = \\ &= \frac{1}{2}(f(\bar{x} + \bar{y}, \bar{x} + \bar{y}) - f(\bar{x}, \bar{x}) - f(\bar{y}, \bar{y})) = \\ &= \frac{1}{2}(f(\bar{x}, \bar{x}) + f(\bar{x}, \bar{y}) + f(\bar{y}, \bar{x}) + f(\bar{y}, \bar{y}) - f(\bar{x}, \bar{x}) - f(\bar{y}, \bar{y})) = \\ &= f(\bar{x}, \bar{y}) \end{aligned}$$

Hence $g = f$ and in particular it is bilinear and symmetric.

2) $\Rightarrow$ 1) It is enough to check that the mapping g is the symmetric bilinear form whose associated quadratic form is ω:

$$\begin{aligned} g(\bar{x}, \bar{x}) &= \frac{1}{2}(\omega(\bar{x} + \bar{x}) - \omega(\bar{x}) - \omega(\bar{x})) = \frac{1}{2}(\omega(2\bar{x}) - 2\omega(\bar{x})) = \\ &= \frac{1}{2}(4\omega(\bar{x}) - 2\omega(\bar{x})) = \omega(\bar{x}) \end{aligned}$$

Remark 3.2 *We deduce from the definition that to any symmetric bilinear form on U we can associate a quadratic form and vice versa. Thus the vector space of all quadratic forms on U is isomorphic with the vector space of symmetric bilinear forms $Bil_S(U)$.*

3.2 Matrices and change of basis.

Let U be a vector space and $B = \{\bar{u}_1, \ldots, \bar{u}_n\}$ a basis. Given any quadratic form $\omega : U \longrightarrow K$ we may consider its associated polar form $f : U \times U \longrightarrow \mathbb{K}$. The matrix expression of f is

$$f(\bar{x}, \bar{y}) = (x)^t F_B(y)$$

Thus, taking into account that $\omega(\bar{x}) = f(\bar{x}, \bar{x})$, the matrix expression of the quadratic form ω is

$$\boxed{\omega(\bar{x}) = (x)^t F_B(x)}$$

Hence the **matrix associated to a quadratic form** relative to a basis B is just the matrix associated to the corresponding polar form. Let us note that, since f is a symmetric bilinear form, the matrix F_B associated to a quadratic form is a symmetric matrix.

If we fix another basis $B' = \{\bar{u}'_1, \ldots, \bar{u}'_n\}$ of U, the relation between the matrices associated to the quadratic form ω relative to the bases B and B' is the same as the relation between the matrices associated to the corresponding polar forms, that is,

$$\boxed{F_{B'} = (M_{BB'})^t F_B M_{BB'}}$$

As a consequence we deduce that

> Any two matrices associated to the same quadratic form on a vector space U are congruent.

The fact that the rank of a matrix is invariant under congruence allows us to introduce the following definition:

Definition 3.3 *Given any quadratic form ω we define its* **rank** *as the rank of any matrix associated to ω.*

3.3 Conjugation.

3.3.1 Conjugate vectors.

Definition 3.4 *Let $\omega : U \longrightarrow \mathbb{K}$ be a quadratic form and let f be its polar form. Two vectors $\bar{x}, \bar{y} \in U$ are said to be* **conjugate** *relative to ω or f if:*

$$f(\bar{x}, \bar{y}) = 0.$$

It is clear that the null vector $\bar{0}$ is conjugate to all vectors in the space:

$$f(\bar{x}, \bar{0}) = 0 \qquad \text{for any } \bar{x} \in U.$$

Definition 3.5 *Let $\omega : U \longrightarrow \mathbb{K}$ be a quadratic form and $\bar{x} \in U$. We say that $\bar{x}$ is* **self-conjugate** *if it is conjugate to itself*

$$\omega(\bar{x}) = 0$$

3.3.2 Conjugate subspaces.

Definition 3.6 *Let $\omega : U \longrightarrow \mathbb{K}$ be a quadratic form. Let A be a subset of U. We call the* **conjugate set of** *A the set of all vectors which are $\omega-$conjugate to all the elements of A, and we denote it by $conj(A)$.*

$$conj(A) = \{\bar{x} \in U |\, f(\bar{x}, \bar{a}) = 0 \ \textit{for every } a \in A\}.$$

Let us see some properties of the conjugate set:

1. *The set $conj(A)$ is a vector subspace of U.*
   Proof: It is clear that $\bar{0} \in conj(A)$, since the null vector is conjugate to any vector. Moreover, fix any $\bar{x}, \bar{y} \in conj(A)$ and $\alpha, \beta \in \mathbb{K}$. Then for any $a \in A$ we have
   $$f(\alpha\bar{x} + \beta\bar{y}, \bar{a}) = \alpha f(\bar{x}, \bar{a}) + \beta f(\bar{y}, \bar{a}) \underset{\uparrow}{=} \bar{0}$$
   $$\bar{x}, \bar{y} \in conj(A) \ \Rightarrow \ f(\bar{x}, \bar{a}) = f(\bar{y}, \bar{a}) = 0$$
   Hence $\alpha\bar{x} + \beta\bar{y} \in conj(A)$.
2. $A \subset B \quad \Rightarrow \quad conj(B) \subset conj(A).$
   Proof:
   $$\bar{x} \in conj(B) \quad \Rightarrow \quad f(\bar{x}, \bar{b}) = 0, \quad \forall \bar{b} \in B \quad \Rightarrow$$
   $$\Rightarrow \quad f(\bar{x}, \bar{a}) = 0, \quad \forall \bar{a} \in A \subset B \quad \Rightarrow \quad \bar{x} \in conj(A).$$
3. $conj(A) = conj(\mathcal{L}(A)).$
   Proof: First we use the preceding property:
   $$A \subset \mathcal{L}(A) \quad \Rightarrow \quad conj(\mathcal{L}(A)) \subset conj(A).$$
   Let us prove the other inclusion. Fix $\bar{x} \in conj(A)$ and $\bar{y} \in \mathcal{L}(A)$. Then
   $$\bar{y} = \sum \alpha^i \bar{a}_i \text{ with } \alpha^i \in \mathbb{K}, \quad \bar{a}_i \in A.$$
   Hence
   $$f(\bar{x}, \bar{y}) = f(\bar{x}, \sum \alpha^i \bar{a}_i) = \sum \alpha^i f(\bar{x}, \bar{a}_i) \underset{\uparrow}{=} 0$$
   $$\bar{x} \in conj(A), \bar{a}_i \in A \quad \Rightarrow \quad f(\bar{x}, \bar{a}_i) = 0.$$
   We deduce that $\bar{x} \in conj(\mathcal{L}(A))$.
4. *The conjugate of any vector space is the same as the conjugate of any of its generating sets.*

3.3.3 The kernel of a quadratic form.

Definition 3.7 *Given any quadratic form $\omega : U \longrightarrow \mathbb{K}$ we define its* **kernel** *as the set of those vectors conjugate to all vectors in the space:*

$$ker(\omega) = \{\bar{x} \in U |\, f(\bar{x}, \bar{y}) = 0, \quad \forall \bar{y} \in U\} = conj(U)$$

where f is the polar form associated to ω.

Let us see some properties of the kernel:

1. *The kernel is a vector subspace.*
2. *Given any basis B of U:*
   $$ker(\omega) = \{\bar{x} \in U |\, F_B(x) = (0)\}.$$
   Proof: It is enough to note that
   $$f(\bar{x}, \bar{y}) = 0, \quad \forall \bar{y} \in U \iff (x)^t F_B(y) = 0, \quad \forall \bar{y} \in U \iff F_B(x) = (0).$$
3. *All vectors in the kernel are self-conjugate.* **The reciprocal is not true.**
4. $dim(ker(\omega)) = dim(U) - rank(\omega).$
   Proof: Recall that the dimension of the subspace of all solutions of a linear system is the dimension of the space minus the number of equations. If we apply this to the subspace
   $$ker(\omega) = \{\bar{x} \in U |\, F_B(x) = (0)\}$$
   we obtain the required relation.

3.3.4 Ordinary and degenerate quadratic forms.

Definition 3.8 *Let $\omega : U \longrightarrow \mathbb{K}$ be a quadratic form on a vector space U:*

- *We say that ω is* **ordinary** *or* **not degenerate** $\iff$ $ker(\omega) = \{\bar{0}\} \iff rank(\omega) = dim(U)$.
- *We say that ω is* **degenerate** $\iff ker(\omega) \neq \{\bar{0}\} \iff rank(\omega) < dim(U)$.

3.4 Diagonalization of a quadratic form.

Definition 3.9 *A quadratic form $\omega : U \longrightarrow \mathbb{K}$ is said to be* **diagonalizable** *if there is a basis of U relative to which the matrix associated to ω is diagonal.*

Remark 3.10 *If the matrix of a quadratic form $\omega : U \longrightarrow \mathbb{K}$ relative to a basis B is diagonal, then its matrix expression is*

$$\omega(\bar{x}) = (x)D\{x\} = d_{11}(x^1)^2 + d_{22}(x^2)^2 + \ldots + d_{nn}(x^n)^2$$

Hence diagonalizing a quadratic form is equivalent to expressing it as a **sum of squares.**

Definition 3.11 *Given any quadratic form $\omega : U \longrightarrow \mathbb{K}$, we call basis of conjugate vectors relative to ω any basis in which every vector is conjugate to all the remaining ones.*

Proposition 3.12 *Let $\omega : U \longrightarrow \mathbb{K}$ be a quadratic form. The basis B is a basis of conjugate vectors if and only if the associated matrix F_B is diagonal.*

Proof: Put $B = \{\bar{u}_1, \ldots, \bar{u}_n\}$. Let us recall that if f is the polar form associated to ω, then its associated matrix is:

$$(F_B)_{ij} = f(\bar{u}_i, \bar{u}_j)$$

Hence

$$\begin{aligned} B \text{ basis of conjugate vectors} &\iff f(\bar{u}_i, \bar{u}_j) = 0, \quad \forall i \neq j \iff \\ &\iff (F_B)_{ij} = 0, \quad \forall i \neq j \iff \\ &\iff (F_B) \text{ is diagonal.} \end{aligned}$$

■

We have seen that a matrix associated to a quadratic form is always symmetric. Moreover, the change of basis transforms an associated matrix into another which is congruent to it. Thus if F_B is the matrix associated to w relative to an arbitrary basis B,

$$\omega \text{ diagonalizable} \iff F_B \text{ diagonalizable by congruence}$$

Taking into account that every symmetric matrix is diagonalizable by congruence we deduce the following theorem:

Theorem 3.13 *Every quadratic form is diagonalizable. Equivalently, given any quadratic form there always exists a basis of conjugate vectors.*

4 Real quadratic forms.

4.1 Canonical expression of a quadratic form.

Let $\omega : U \longrightarrow \mathbb{R}$ be a quadratic form. We have just proved that ω is diagonalizable. In particular there exists a basis B relative to which the associated matrix has the form:

$$F_B = \left(\begin{array}{c|c|c} I_p & \Omega & \Omega \\ \hline \Omega & -I_q & \Omega \\ \hline \Omega & \Omega & \Omega \end{array}\right). \qquad (*)$$

Definition 4.1 *We call the* **signature of the quadratic form** *ω the pair or natural numbers (p, q) where p is the number of positive elements in its diagonal form and q the number of negative elements:*

$$Sig(\omega) = (p, q).$$

These two numbers satisfy $p + q = rank(\omega)$.

In the following theorem, called **Sylvester's law of inertia** it is proved that the preceding definition makes sense i. e. the numbers (p, q) do not depend on the basis relative to which it is diagonalized.

Theorem 4.2 (Sylvester's law of inertia) *The signature of any quadratic form $\omega : U \longrightarrow \mathbb{R}$ is invariant, that is, it does not depend on the basis.*

Proof: Fix two basis of U

$$B = \{\bar{u}_1, \ldots, \bar{u}_n\}, \qquad B' = \{\bar{u}'_1, \ldots, u'_n\},$$

relative to which the matrix associated to ω is diagonal (and has the form (*)).

Assume that the signatures relative to the bases B and B' are (p, q) and (p', q') respectively. Put

$$\begin{aligned} U_1 &= \mathcal{L}\{\bar{u}_1, \ldots, \bar{u}_p\} \\ U_2 &= \mathcal{L}\{\bar{u}'_{p'+1}, \ldots, \bar{u}'_n\} \end{aligned}$$

If $\bar{x} \in U_1 \cap U_2$ is a nonzero vector

$$\begin{array}{lll} \bar{x} \in U_1 & \Rightarrow & \omega(\bar{x}) > 0 \\ \bar{x} \in U_2 & \Rightarrow & \omega(\bar{x}) \leq 0 \end{array}$$

hence $U_1 \cap U_2 = \{\bar{0}\}$. Consequently

$$dim(U_1)+dim(U_2) = dim(U_1+U_2)+dim(U_1 \cap U_2) \quad \Rightarrow \quad p+(n-p') \leq n \quad \Rightarrow \quad p \leq p'$$

If we repeat the argument with the roles of p and p' reversed we obtain $p' \leq p$. Hence

$$p = p' \quad \Rightarrow \quad q = n - p = n - p' = q'.$$

4.2 Classification of quadratic forms.

Definition 4.3 *Let $\omega : U \longrightarrow \mathbb{R}$ be a quadratic form:*

- ω *is* **positive definite** $\iff$ $\omega(\bar{x}) > 0, \quad \forall \bar{x} \neq \bar{0}.$
- ω *is* **positive semidefinite** $\iff$ *It is not positive definite and* $\omega(\bar{x}) \geq 0, \quad \forall \bar{x} \neq \bar{0}.$
- ω *is* **negative definite** $\iff$ $\omega(\bar{x}) < 0, \quad \forall \bar{x} \neq \bar{0}.$
- ω *es* **negative semidefinite** $\iff$ *It is not negative definite and* $\omega(\bar{x}) \leq 0, \quad \forall \bar{x} \neq \bar{0}.$
- ω *is* **indefinite** $\iff$ $\exists \bar{x}, \bar{y} \neq 0$ *with* $\omega(\bar{x}) > 0, \quad \omega(\bar{y}) < 0.$

All these definitions can be given for real symmetric matrices, if we take into account that every such matrix determines a quadratic form:

Definition 4.4 *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ be a real symmetric matrix:*

- A *is* **positive definite** $\iff$ A *is congruent with* I.
- A *es* **positive semidefinite** $\iff$ A *is congruent with* $\left(\begin{array}{c|c} I & \Omega \\ \hline \Omega & \Omega \end{array}\right)$.
- A *es* **negative definite** $\iff$ A *is congruent with* $-I$.
- A *is* **negative semidefinite** $\iff$ A *is congruent with* $\left(\begin{array}{c|c} -I & \Omega \\ \hline \Omega & \Omega \end{array}\right)$.
- A *es* **indefinite** $\iff$ A *is congruent with* $\left(\begin{array}{c|c|c} I & \Omega & \Omega \\ \hline \Omega & -I & \Omega \\ \hline \Omega & \Omega & \Omega \end{array}\right)$.

Proposition 4.5 *Let $\omega : U \longrightarrow \mathbb{R}$ be a quadratic form on an n-dimensional vector space U.*

- ω *is* **positive definite** $\iff$ $Sig(\omega) = (n, 0)$.
- ω *es* **positive semidefinite** $\iff$ $Sig(\omega) = (p, 0)$ *with* $p < n$.
- ω *is* **negative definite** $\iff$ $Sig(\omega) = (0, n)$.
- ω *is* **negative semidefinite** $\iff$ $Sig(\omega) = (0, q)$ *with* $q < n$.
- ω *es* **indefinite** $\iff$ $Sig(\omega) = (p, q)$ *with* $p > 0$, $q > 0$.

Proof: This is easy to check if we take into account that if $Sig(\omega) = (p, q)$ then the expression of ω relative to a convenient basis is:

$$\omega(\bar{x}) = (x^1)^2 + (x^2)^2 + \ldots + (x^p)^2 - (x^{p+1})^2 - (x^{p+2})^2 - \ldots - (x^{p+q})^2.$$

■

Corollary 4.6 (Description of the set of self-conjugate vectors.) *Let $\omega : U \longrightarrow \mathbb{R}$ be a quadratic form on an n-dimensional vector space U. Let $Self(\omega)$ be the set of all self-conjugate vectors. Then*

1. *If ω is positive definite or negative definite then $Self(\omega) = \{\vec{0}\}$.*
2. *If ω is positive semidefinite or negative semidefinite then $Self(\omega) = ker(\omega)$.*
3. *If ω is indefinite, then*

(a) If $rank(\omega) = 2$ then $Self(\omega)$ is the union of two hiperplanes whose intersection is $ker(w)$.

(b) If $rank(\omega) > 2$ then $Self(\omega)$ is a quadric whose equation cannot be expressed as the product of the equations of two hyperplanes.

Proof: Let us recall that

$$Self(\omega) = \{\vec{u} \in U | \omega(\vec{u}) = 0\}.$$

If the quadratic form is positive (or negative) definite then $\omega(\vec{u}) \neq 0$ whenever $\vec{u} \neq \vec{0}$. Hence $Self(\omega) = \{\vec{0}\}$.

If the quadratic form is positive semidefinite then we know that the associated matrix F_B relative to a convenient basis is B is diagonal with $k = n - rank(w)$ ones and $rank(w)$ zeros on the diagonal. In coordinates relative to such a basis

$$w((x_1, x_2, \dots, x_n)_B) = x_1^2 + x_2^2 + \dots + x_k^2$$

being $n - k = rank(w)$.

Hence

$$(x_1, x_2, \dots, x_n)_B \in Self(\omega) \iff x_1^2 + x_2^2 + \dots + x_k^2 = 0 \iff x_1 = x_2 = \dots = x_k = 0.$$

On the other hand a vector $(x_1, x_2, \dots, x_n)_B$ is in the kernel if it satisfies

$$(x_1, x_2, \dots, x_n)_B F_B = (0, 0, \dots, 0) \iff x_1 = x_2 = \dots = x_k = 0$$

We see that indeed $Self(\omega) = ker(\omega)$. In an analogous way one can analyze the case where the quadratic form is negative semidefinite.

Finally if it is indefinite, once diagonalized there will be both positive and negative terms on the diagonal:

- If $rank(\omega) = 2$ then in coordinates relative to a convenient basis B,

$$w((x_1, x_2, \dots, x_n)_B) = x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2).$$

Thus

$$w((x_1, x_2, \dots, x_n)_B) = 0 \iff x_1 - x_2 = 0 \text{ or } x_1 + x_2 = 0$$

and we see that indeed, the set of self-conjugate vectors is the union of two hyperplanes, one given by the equation $x_1 - x_2 = 0$ and the other given by the equation $x_1 + x_2 = 0$. Furthermore the kernel is the set of vectors satisfying the equations $x_1 = x_2 = 0$, which is exactly the intersection of the two hyperplanes.

- If $rank(\omega) > 2$ then in coordinates relative to a convenient basis B,

$$w((x_1, x_2, \dots, x_n)_B) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2$$

with $p, q > 1$ and $p+q > 2$ and there is no way to decompose or simplify the equation

$$x_1^2 + \dots + x_p^2 + x_{p+1}^2 - \dots - x_{p+q}^2 = 0.$$

■

Proposition 4.7 *The determinants of any two congruent matrices always have the same sign.*

Proof: If A and B are congruent there exists a regular matrix C with

$$A = CBC^t \quad \Rightarrow \quad |A| = |C||B||C^t| = |C|^2|B| \quad \Rightarrow \quad sign(|A|) = sign(|B|).$$

■

Theorem 4.8 (Sylvester's criterion) *Let $\omega : U \longrightarrow \mathbb{R}$ be a quadratic form and let F be its associated matrix relative to a basis $B = \{\bar{u}_1, \dots, \bar{u}_n\}$. We put*

$$F_1 = (f_{11}), \quad F_2 = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}, \quad F_3 = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}, \quad etc\dots$$

Then

1. *ω is* **positive definite** $\iff$ *$|F_i| > 0$, for every $i = 1, \dots, n$.*
2. *ω is* **negative definite** $\iff$ *$(-1)^i |F_i| > 0$, for every $i = 1, \dots, n$.*

Proof:

1. $\Longrightarrow$: If ω is positive definite on the whole space U, it is also positive definite when restricted to any subspace

$$U_i = \mathcal{L}\{\bar{u}_1, \dots, \bar{u}_i\}.$$

The matrix of the restriction of ω to U_i is F_i. Because this restriction is positive definite, F_i is congruent with Id, and in particular its determinant is positive.

$\Longleftarrow$: Let us assume that $|F_i| > 0$, for every $i = 1, \dots, n$. We will show by induction that ω is positive definite:

- For $n = 1$ it is clear.

- Assume the result true for quadratic forms on vector spaces of dimension $\leq n - 1$ and let us see that it is true for dimension n.

By our induction hypothesis we know that ω is positive definite on the vector subspace $U_1 = \mathcal{L}\{\bar{u}_1, \dots, \bar{u}_{n-1}\}$. Hence there exists a basis B'_1 of U_1 relative to which the matrix associated to the restriction of ω to U_1 is the identity. Let us consider the basis

$$B' = B'_1 \cup \{\bar{u}\}$$

The matrix associated to ω relative to this basis is:

$$\left(\begin{array}{ccc|c} & & & a_1 \\ & I & & \vdots \\ & & & a_{n-1} \\ \hline a_1 & \dots & a_{n-1} & a_n \end{array}\right)$$

If we diagonalize by congruence we obtain a matrix which is congruent to the matrix F of ω relative to the initial basis:

$$C = \left(\begin{array}{ccc|c} & & & 0 \\ & I & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & b \end{array}\right)$$

Since F and C are congruent, their determinants have the same sign. Thus $|C| > 0$, $b > 0$ and the signature of ω is $(n, 0)$. Hence ω is indeed positive definite.

2. To show that ω is **negative definite** $\iff (-1)^i |F_i| > 0$, for every $i = 1, \dots, n$, take into account that

$$\omega \text{ is negative definite} \iff -\omega \text{ is positive definite.}$$

Now it suffices to apply the previous item.