Complete description of a conic from its equation

Classification

$\lvert T\rvert > 0$ (Elliptic type)	A definite: IMAGINARY ELLIPSE A indefinite, nondegenerate: ELLIPSE $\lvert A\rvert = 0$: TWO INTERSECTING IMAGINARY LINES
$\lvert T\rvert < 0$ (Hyperbolic type)	$\lvert A\rvert \neq 0$: HYPERBOLA $\lvert A\rvert = 0$: TWO INTERSECTING REAL LINES
$\lvert T\rvert = 0$ (Parabolic type)	$\lvert A\rvert \neq 0$: PARABOLA $\operatorname{rank}(A) = 2$, A semidefinite: TWO PARALLEL IMAGINARY LINES $\operatorname{rank}(A) = 2$, A indefinite: TWO PARALLEL REAL LINES $\operatorname{rank}(A) = 1$: DOUBLE LINE

Notable points and lines that can be directly obtained from the matrix A

CENTER: (r, s) with $\boxed{A\begin{pmatrix} r \\ s \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}}$ where h is arbitrary.

ASYMPTOTIC DIRECTIONS : (p, q) with $\boxed{\begin{pmatrix} p & q \end{pmatrix} T \begin{pmatrix} p \\ q \end{pmatrix} = 0}$.

ASYMPTOTES: For each asymptotic direction (p, q), the polar line $\boxed{\begin{pmatrix} p & q & 0 \end{pmatrix} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0}$

AXES: Compute the eigenvalues λ_1, λ_2 of T as roots of $p(\lambda) = |T - \lambda Id|$. For those which are not zero, find the corresponding eigenvectors $\vec{u}_i = (x_i, y_i)$ by solving the system $(T - \lambda_i Id)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The axes are their polar lines: $\boxed{\underbrace{\begin{pmatrix} x_i & y_i & 0 \end{pmatrix}}_{\text{eigenvector}} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0}$

VERTICES =Axes$\cap$conic . Solve the systems formed by the equations of the conic and each one of the axes.

Note: For both the ellipse and the hyperbola, if we follow the criterion for the order of the eigenvalues indicated below, the focal axis is the polar line of the eigenvector associated with λ_2.

Reduced equations, foci, directrices and eccentricity

Reduced equation of the ellipse:

Reduced equation $\boxed{\lambda_1 x'^2 + \lambda_2 y'^2 + d = 0}$ where λ_1, λ_2 are the eigenvalues of T (they have the same sign), sorted in such a way that $|\lambda_1| \leq |\lambda_2|$, and $d = \frac{\det(A)}{\det(T)}$.

Equations of the change of reference system: $\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \text{center} \end{pmatrix} + \underbrace{\begin{pmatrix} \text{eigenvec}_1 & \text{eigenvec}_2 \end{pmatrix}}_{\text{normalized}} \begin{pmatrix} x' \\ y' \end{pmatrix}}$

Canonical equation $\boxed{\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1}$ (this is directly obtained from the reduced equation).

Foci relative to the new reference system: $F_1' = (c, 0)$ and $F_2' = (-c, 0)$, with $c = +\sqrt{a^2 - b^2}$.

Foci relative to the initial reference system: Apply the change of reference system to F_1' and F_2'.

Directrices: Polar lines of the foci $F_i = (s_i, t_i)$: $\boxed{\underbrace{\begin{pmatrix} s_i & t_i & 1 \end{pmatrix}}_{\text{focus}} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0}$

Eccentricity: $\boxed{e = \frac{c}{a} < 1}$.

Reduced equation of the hyperbola:

Reduced equation $\boxed{\lambda_1 x'^2 + \lambda_2 y'^2 + d = 0}$ with λ_1, λ_2 eigenvalues of T, sorted in such a way that $\text{sign}(\lambda_1) = \text{sign}(\det(A))$ and $d = \frac{\det(A)}{det(T)}$.

Equations of the change of reference system: $\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \text{center} \end{pmatrix} + \underbrace{\begin{pmatrix} \text{eigenvec}_1 & \text{eigenvec}_2 \end{pmatrix}}_{\text{normalized}} \begin{pmatrix} x' \\ y' \end{pmatrix}}$.

Canonical equation $\boxed{\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1}$ (this is directly obtained from the reduced equation).

Foci relative to the new reference system: $F_1' = (c, 0)$ and $F_2' = (-c, 0)$, with $c = +\sqrt{a^2 + b^2}$.

Foci relative to the initial reference system: Apply the change of reference system to F_1' and F_2'.

Directrices: Polar lines of the foci $F_i = (s_i, t_i)$: $\boxed{\underbrace{\begin{pmatrix} s_i & t_i & 1 \end{pmatrix}}_{\text{focus}} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0}$

Eccentricity: $\boxed{e = \frac{c}{a} > 1}$.

Reduced equation of the parabola:

Reduced equation $\boxed{\lambda_1 x'^2 - 2dy' = 0}$ with λ_1 nonzero eigenvalue of T and $d = \pm\sqrt{\frac{-\det(A)}{\lambda_1}}$ (choose the sign of d so it is different from that of λ_1).

Equations of the change of reference system: $\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \text{vertex} \end{pmatrix} + \underbrace{\begin{pmatrix} \text{eigenvec}_1 & \text{eigenvec}_2 \end{pmatrix}}_{\text{normalized}} \begin{pmatrix} x' \\ y' \end{pmatrix}}$.

We must choose the orientation of the eigenvector associated with 0 in such a way that the signs of $\begin{pmatrix} a_{13} & a_{23} \end{pmatrix} \begin{pmatrix} \text{eigenvec}_2 \end{pmatrix}$ and λ_1 are different.

Canonical equation $\boxed{x'^2 = 2py'}$ (this is directly obtained from the reduced equation).

Focus relative to the new reference system: $F' = (0, p/2)$.

Focus relative to the initial reference system: Apply the change of reference system to F'.

Directrix: Polar line of the focus $F = (s, t)$: $\boxed{\underbrace{\begin{pmatrix} s & t & 1 \end{pmatrix}}_{\text{focus}} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0}$

Eccentricity: $\boxed{e = 1}$.

Polar line of a point with respect to the conic

Polar line of a point: $\boxed{\begin{pmatrix} a & b & 1 \end{pmatrix} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0}$. If the point is on the conic then the polar line is the tangent line to the conic through the point.

Polar line of a vector or direction: $\boxed{\begin{pmatrix} p & q & 0 \end{pmatrix} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0}$. If (p, q) is an asymptotic direction then the polar line is an asymptote.