# Equation of a conic from certain data

**I. General remarks:**

1. The center or an axis can be used to duplicate the avaliable information by symmetry. **Ex. 11d., 11f., 12, 15, 16, XIII**.
2. If a vertex is known, it may be useful to note that the axis is perpendicular to the tangent line through the vertex. **Ex. 11g, 11i, 14.**
3. If two foci are known, the most efficient way to obtain the equation of the conic is using the description of the conic as a locus. **Ex. 11c. 17.XI.**
4. If a focus and its associated directrix are known, it will be useful to use its definition, that is, the quotient of the distances from an arbitrary point on the conic to the focus and from the same point to the directrix is constant and equal to the eccentricity.
5. If it is known that the conic is a parabola, then the determinant of the matrix of quadratic terms must be null: $|T| = 0$. **Ex. 11e,11i, XIII.**

**II. Method (1): Find its associated matrix $A$ directly.**

We know that the matrix associated with a conic is a symmetric one. So it depends on 6 parameters:

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}.$$

We just have to use the known data on the conic to impose conditions on this matrix, gradually obtaining equations that allow us to obtain all its coefficients. Since proportional equations define the same geometric object, it will be enough if we manage to write five of the parameters as multiples of the remaining one, so at the end we only have to divide by the latter. In other words, in general it will suffice to obtain five independent equations.

The data that we will have at hand most frequently, and its translation into equations involving the coefficients of the matrix, are as follows:

1. A point $(x_0, y_0)$ on the conic. We then impose that $(x_0, y_0, 1)A(x_0, y_0, 1)^t = 0$ and we obtain ONE equation. **Ex. 11e**.
2. We know the center $(x_0, y_0)$ of the conic. We then impose that $(x_0, y_0, 1)A = (0, 0, h)$ and we obtain TWO equations. **Ex. 11b**.
3. We know a tangent $px + qy + r = 0$ and the point of tangency $(x_0, y_0)$. We impose that the tangent line at the indicated point is the one given:

   $$(x_0, y_0, 1)A(x, y, 1)^t = 0 \quad \equiv \quad px + qy + r = 0.$$

   We will use the fact that two equations define the same line if and only if their coefficients are proportional. Hence TWO equations are obtained. **Ex. XIV**.
4. We know a tangent line $px + qy + r = 0$ but NOT the point of tangency. We impose that the intersection of the tangent and the conic is a single point. This means that the discriminant of the quadratic equation obtained by combining the equation of the line and that of the conic is zero. ONE equation is obtained. **Ex. XIV.**
5. We know the polar line $px + qy + r = 0$ of a point $(x_0, y_0)$. We impose that the polar line of the indicated point is the given line:

   $$(x_0, y_0, 1)A(x, y, 1)^t = 0 \quad \equiv \quad px + qy + r = 0.$$

   Again it will be used that two equations define the same line if their coefficients are proportional. TWO equations are obtained. **Ex. 11b,11e,11i**.
6. We know an axis $px + qy + r = 0$. We use the fact that the axis is the polar line of its normal vector:

   $$(p, q, 0)A(x, y, 1)^t = 0 \quad \equiv \quad px + qy + r = 0.$$

   Again it will be used that two equations define the same line if their coefficients are proportional. TWO equations are obtained. **Ex. 11b**.
7. We know an asymptote $px + qy + r = 0$. We use that the asymptote is the tangent line at its point of infinity (the one given by its direction vector):

   $$(q, -p, 0)A(x, y, 1)^t = 0 \quad \equiv \quad px + qy + r = 0.$$

   Again it will be used that two equations define the same line if their coefficients are proportional. TWO equations are obtained. **Ex VIII**

**II. Method (2): Pencils of conics.**

The idea is to use all the pieces of information we have except one of them, to define a pencil of conics which gives the equation of the required conic up to a single unknown (or two proportional parameters).

To generate the pencil, we need two conics fulfilling all but one of the conditions in the statement. Normally (but not always) one will choose conics formed by pairs of lines.

The condition that we have not used for the definition of the pencil is finally invoked to find the value of the remaining unknown. This condition is incorporated following the same indications that we have seen in Method (1)).

The most typical pencils are:

1. Pencil of all conics containing four given points.
2. Pencil given two points, a tangent line and the corresponding point of tangency.
3. Pencil given two tangent lines and their corresponding points of tangency **Ex. 11e, 11g, 12, 16, XIV.**.
4. Pencil given an asymptote, a tangent line and the corresponding point of tangency **Ex. IX**.
5. Pencil given two asymptotes **Ex. 11d, 14, 15, VIII**.
6. Special pencils adapted to the data in the problem **Ex. 8iv, V**.