3. Linear maps.

1 Definition and properties.

Definition 1.1 *Let U and V be two vector spaces over the field $\mathbb{K}$. We say that $f : U \longrightarrow V$ is a* **linear map** *or* **homomorphism** *if it satisfies:*

1. $f(\bar{x} + \bar{x}') = f(\bar{x}) + f(\bar{x}')$, *for any* $\bar{x}, \bar{x}' \in U$.
2. $f(\alpha\bar{x}) = \alpha f(\bar{x})$, *for any* $\alpha \in \mathbb{K}$, $\bar{x} \in U$.

or equivalently:

3. $f(\alpha\bar{x} + \beta\bar{x}') = \alpha f(\bar{x}) + \beta f(\bar{x}')$, *for any* $\alpha \in \mathbb{K}$, $\bar{x}, \bar{x}' \in U$.

Remark 1.2 *Let us see the equivalence between conditions* 1., 2. *and condition* 3.*:*

1., 2. $\Longrightarrow$ 3.*:*

Given $\alpha \in \mathbb{K}$, $\bar{x}, \bar{x}' \in U$ *we have:*

$$f(\alpha\bar{x} + \beta\bar{x}') \underset{(1.)}{\overset{\uparrow}{=}} f(\alpha\bar{x}) + f(\beta\bar{x}') \underset{(2.)}{\overset{\uparrow}{=}} \alpha f(\bar{x}) + \beta f(\bar{x}').$$

3. $\Longrightarrow$ 1., 2.*:*

If we apply property 3. *for* $\alpha = \beta = 1$ *and any* $\bar{x}, \bar{x}' \in U$, *we obtain property* 1.

If we apply property 3. *for* $\beta = 0$, $\bar{x}' = \bar{0}$ *and any* $\alpha \in \mathbb{K}$, $\bar{x} \in U$, *we obtain property* 2..

We next collect some properties of linear maps:

a) $f(-\bar{x}) = -f(\bar{x})$.
Proof: We apply property 2. from the definition:

$$f(-\bar{x}) = f((-1)\bar{x}) = (-1)f(\bar{x}) = -f(\bar{x}).$$

b) $f(\bar{x} - \bar{x}') = f(\bar{x}) - f(\bar{x}')$.
Proof: It is sufficient to apply property 3. from the definition with $\alpha = 1$ amd $\beta = -1$.

c) $f(\bar{0}) = \bar{0}$.
Applying the previous property:

$$f(\bar{0}) = f(\bar{0} - \bar{0}) = f(\bar{0}) - f(\bar{0}) = \bar{0}.$$

d) *Main property:*

A linear map is completely determined by the images of the elements of any basis

Proof: If $B = \{\bar{u}_1, \dots, \bar{u}_m\}$ is a basis of U, then any other vector $\bar{x} \in U$ is uniquely expressed as a combination linear of the elements of B. Applying the properties of the linear map definition we have:

$$f(\bar{x}) = f(x^1\bar{u}_1 + \dots + x^m\bar{u}_m) = x^1 f(\bar{u}_1) + \dots + x^m f(\bar{u}_m).$$

We will sometimes use the abbreviated notation

$$f(\bar{x}) = f(x^i\bar{u}_i) = x^i f(\bar{u}_i).$$

e) *Let* $A = \{\bar{a}_1, \dots, \bar{a}_p\}$ *be a set of vectors and let* $f(A) = \{f(\bar{a}_1), \dots, f(\bar{a}_p)\}$ *the image set of* A *by* f.

e.1) *If $f(A)$ is a set of independent vectors then the vectors of A are independent.*
Proof: Consider a linear combination of the vectors in A which is equal to $\bar{0}$:

$$\alpha^1\bar{a}_1 + \dots + \alpha^p\bar{a}_p = \bar{0}.$$

We apply f to both members of the equality:

$$f(\alpha^1\bar{a}_1 + \dots + \alpha^p\bar{a}_p) = f(\bar{0}) \quad\Rightarrow\quad \alpha^1 f(\bar{a}_1) + \dots + \alpha^p f(\bar{a}_p) = \bar{0}$$

Since $f(A)$ is an independent set, $\alpha^1 = \dots = \alpha^p = 0$ and we deduce that A is an independent set.

e.2) *If the vectors of A are dependent, then $f(A)$ is a dependent set.*
Proof: If the vectors of A are dependent then one of them is a linear combination of the others. Suppose it is $\bar{a}_1$:

$$\bar{a}_1 = \alpha^2\bar{a}_2 + \dots + \alpha^p\bar{a}_p.$$

We apply f to both members of the equality:

$$f(\bar{a}_1) = f(\alpha^2\bar{a}_2 + \dots + \alpha^p\bar{a}_p) \quad\Rightarrow\quad f(\bar{a}_1) = \alpha^2 f(\bar{a}_2) + \dots + \alpha^p f(\bar{a}_p).$$

We see that one of the vectors in $f(A)$ is a linear combination of the others. Therefore $f(A)$ is a dependent set.

Remark: None of the converses of these two results is true. It can happen that A is independent and $f(A)$ is dependent. For example if we take the linear map

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}; \qquad f(x, y) = x + y.$$

and $A = \{(1, 0), (1, 1)\}$, then $f(A) = \{1, 2\}$ and we see that A is independent but $f(A)$ is dependent.

f) If $f : U \longrightarrow V$ is a linear map and S a vector subspace of U, then the **restriction of f to S** is the map:

$$f_s : S \longrightarrow V, \qquad f_s(\bar{x}) = f(\bar{x}).$$

2 Matrix representation of linear maps.

Let U and V be two vector spaces over $\mathbb{K}$. Suppose B_1 and B_2 are bases of U and V respectively:

$$B_1 = \{\bar{u}_1, \dots, \bar{u}_m\}; \qquad B_2 = \{\bar{v}_1, \dots, \bar{v}_n\}.$$

The image of each one of the vectors in B_1 is in V, so it can be uniquely expressed as a linear combination of the vectors in B_2:

$$\begin{array}{c} f(\bar{u}_1) = a_{11}\,\bar{v}_1 + a_{21}\,\bar{v}_2 + \dots + a_{n1}\,\bar{v}_n \\ f(\bar{u}_2) = a_{12}\,\bar{v}_1 + a_{22}\,\bar{v}_2 + \dots + a_{n2}\,\bar{v}_n \\ \vdots \\ f(\bar{u}_m) = a_{1m}\,\bar{v}_1 + a_{2m}\,\bar{v}_2 + \dots + a_{nm}\,\bar{v}_n \end{array}$$

This corresponds to the following matrix expression:

$$\{\, f(\bar{u}_1) \quad f(\bar{u}_2) \quad \dots \quad f(\bar{u}_m) \,\} = \{\, \bar{v}_1 \quad \bar{v}_2 \quad \dots \quad \bar{v}_n \,\} \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}}_{F_{B_2B_1}},$$

or, equivalently,

$$(f(\bar{u}_j)) = (\bar{v}_i)F_{B_2B_1}.$$

The matrix $F_{B_2B_1}$ is called the **matrix associated to the linear map** f **with respect to the bases** B_1 **and** B_2**.**

This matrix allows us to compute the image of any vector by means of a matrix product. We denote:

$(x^1, \dots, x^m) \ \rightarrow$ coordinates of a vector $\bar{x} \in U$ with respect to the basis B_1.
$(y^1, \dots, y^n) \ \rightarrow$ coordinates of the image vector $f(\bar{x}) \in V$ with respect to the basis B_2.

Then:

$$\boxed{\begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^m \end{pmatrix}} \iff \boxed{(y^i) = F_{B_2B_1}(x^j)}$$

Proof: We have:

$$\left.\begin{array}{l} f(\bar{x}) = f((x^j)(\bar{u}_j)) = f(\bar{u}_j)(x^j) = (\bar{v}_i)F_{B_2B_1}(x^j) \\ f(\bar{x}) = (\bar{v}_i)(y^i) \end{array}\right\} \Rightarrow \quad (y^i) = F_{B_2B_1}(x^j).$$

3 Change of basis.

We consider a linear map $f : U \longrightarrow V$ between two vector spaces with finite dimension. Suppose we have the following bases:

$$\begin{array}{l} B_1 = \{\bar{u}_1, \dots, \bar{u}_m\} \\ B'_1 = \{\bar{u}'_1, \dots, \bar{u}'_m\} \end{array} \text{ bases of } U; \qquad \begin{array}{l} B_2 = \{\bar{v}_1, \dots, \bar{v}_n\} \\ B'_2 = \{\bar{v}'_1, \dots, \bar{v}'_n\} \end{array} \text{ bases of } V.$$

We will denote the coordinates of a vector $\bar{x}$ and its image $\bar{y}$ in each of the bases as follows:

Coordinates of $\bar{x}$.		Coordinates of $\bar{y} = f(\bar{x})$.	
$(x^1, \dots, x^m)$	w.r.t. the basis B_1	$(y^1, \dots, y^n)$	w.r.t. the basis B_2
$(x'^1, \dots, x'^m)$	w.r.t. the basis B'_1	$(y'^1, \dots, y'^n)$	w.r.t. the basis B'_2

These different bases and coordinates are related by the following formulas:

$$\begin{array}{l|l} (\bar{u}') = (\bar{u})M_{B_1B'_1} & (\bar{v}') = (\bar{v})M_{B_2B'_2} \\ (x) = M_{B_1B'_1}(x') & (y) = M_{B_2B'_2}(y') \end{array}$$

We also know that we can write the matrix expression of the map f with respect to the bases B_1, B_2 or B'_1, B'_2:

$$(y) = F_{B_2B_1}(x) \ \big| \ (y') = F_{B'_2B'_1}(x')$$

Let us see how the matrices associated to f with respect to the bases B_1, B_2 and B'_1, B'_2 are related:

$$\begin{array}{ll} (y) = F_{B_2B_1}(x) & \Rightarrow \ M_{B_2B'_2}(y') = F_{B_2B_1}M_{B_1B'_1}(x') \ \Rightarrow \\ & \Rightarrow \ (y') = (M_{B_2B'_2})^{-1}F_{B_2B_1}(M_{B_1B'_1})(x') \end{array}$$

and therefore:

$$F_{B'_2B'_1} = (M_{B_2B'_2})^{-1}F_{B_2B_1}M_{B_1B'_1}, \quad \text{or} \quad \boxed{F_{B'_2B'_1} = M_{B'_2B_2}F_{B_2B_1}M_{B_1B'_1}}$$

Some remarks about the change of basis:

1. Two matrices associated to the same linear map with respect to different bases are equivalent.
2. As a mnemonic, we note that the indices corresponding to the same base are adjacent in the formulas:

$$F_{B'_2B'_1} = M_{B'_2\boxed{B_2}}F_{\boxed{B_2}\boxed{B_1}}M_{\boxed{B_1}B'_1}$$

Also, with this notation, the role of each matrix is explicitly indicated. That is, we want to express the matrix $F_{B'_2B'_1}$ which, when multiplied by coordinates in the base B'_1, returns the image in the base B'_2, in terms of the matrix $F_{B_2B_1}$, which, when multiplied by coordinates in the base B_1, returns the image in base B_2. Sos:

- through $M_{B_1B'_1}$ we turn the coordinates in the base B'_1 into coordinates in the base B_1.

- through $F_{B_2B_1}$ we obtain the coordinates of the image in the base B_2 from the coordinates in the base B_1 obtained in the previous step.

- through $M_{B'_2B_2}$ we turn the coordinates of the image in the base B_2 into coordinates in the base B'_2.

4 Kernel and image of a linear map.

Definition 4.1 *Given a linear map $f : U \longrightarrow V$, the set of vectors whose image by f is $\bar{0}$ is called the* **kernel of** *f:*

$$ker(f) = \{\bar{x} \in U |\, f(\bar{x}) = \bar{0}\}.$$

Proposition 4.2 *The kernel of a linear map is a vector subspace of the initial space.*

Proof: Let us check that it satisfies the definition of a vector subspace:

- First of all, it is nonempty since $f(\bar{0}) = \bar{0}$, so at least $\bar{0} \in ker(f)$.
- Fix $\bar{x}, \bar{x}' \in ker(f)$ and $\alpha, \beta \in \mathbb{K}$ and let us see that $\alpha\bar{x} + \beta\bar{x}' \in ker(f)$:

$$f(\alpha\bar{x} + \beta\bar{x}') = \alpha f(\bar{x}) + \beta f(\bar{x}') \underset{\substack{\uparrow \\ \bar{x}, \bar{x}' \in ker(f)}}{=} \alpha\bar{0} + \beta\bar{0} = \bar{0}.$$

and hence $\alpha\bar{x} + \beta\bar{x}' \in ker(f)$. ∎

If we know the matrix associated to the linear map f with respect to two bases B_1 and B_2, of U and V respectively, then the vectors of the kernel are those whose coordinates with respect to the base B_1 satisfy the equation

$$F_{B_2B_1}\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Definition 4.3 *Given a linear map $f : U \longrightarrow V$ we call the* **image of** *f the set of images by f of all vectors in U:*

$$im(f) = \{\bar{y} \in V |\, \bar{y} = f(\bar{x}),\, \bar{x} \in U\}.$$

Proposition 4.4 *The image of a linear map is a vector subspace of the final space.*

Proof: Let us check that it satisfies the definition of a vector subspace:

- First of all, it is not empty since $f(\bar{0}) = \bar{0}$, thus at least $\bar{0} \in im(f)$.
- Fix $\bar{y}, \bar{y}' \in im(f)$ and $\alpha, \beta \in \mathbb{K}$ and let us see that $\alpha\bar{y} + \beta\bar{y}' \in im(f)$:

$$\left.\begin{array}{lll} \bar{y} \in im(f) & \Rightarrow & \bar{y} = f(\bar{x}), \quad \bar{x} \in U \\ \bar{y}' \in im(f) & \Rightarrow & \bar{y}' = f(\bar{x}'), \quad \bar{x}' \in U \end{array}\right\} \Rightarrow \alpha\bar{y} + \beta\bar{y}' = f(\alpha\bar{x} + \beta\bar{x}')$$

where $\alpha\bar{x} + \beta\bar{x}' \in U$. We deduce that $\alpha\bar{y} + \beta\bar{y}' \in im(f)$. ∎

Proposition 4.5 *If $A = \{\bar{a}_1, \ldots, \bar{a}_k\}$ is a spanning set of U, then $f(A) = \{f(\bar{a}_1), \ldots, f(\bar{a}_k)\}$ is a spanning set of $im(f)$.*

Proof: First of all it is clear that $\mathcal{L}(f(A)) \subset imf$, since any linearcombination of elements in $f(A)$ is the image of the corresponding linear combination of the elements in A.

Conversely, let $\bar{y}$ be in $im(f)$. So there exists $\bar{x} \in U$ with $f(\bar{x}) = \bar{y}$. Since A is a spanning set of U,

$$\bar{x} = \alpha^1\bar{a}_1 + \ldots \alpha^k\bar{a}_k.$$

Applying f we obtain:

$$f(\bar{x}) = f(\alpha^1\bar{a}_1 + \ldots \alpha^k\bar{a}_k) \quad \Rightarrow \quad \bar{y} = f(\bar{x}) = \alpha^1 f(\bar{a}_1) + \ldots \alpha^k f(\bar{a}_k).$$

so $\bar{y}$ can be written as a linear combination of the elements of $f(A)$. ∎

Let B_1 and B_2 be bases of U_1 and U_2 respectively. As a consequence of this Proposition, the image of f is generated by the images of the vectors in B_1. In particular, if $F_{B_2B_1}$ is the matrix associated to f, **the image vectors expressed in coordinates with respect to the base B_2 are generated by the columns of the matrix $F_{B_2B_1}$.**

We note that **these columns do not have to be independent**, since as we saw before, although B_1 is an independent set, $f(B_1)$ does not have to be. Therefore if we want a basis of the image, we need to remove the columns of $F_{B_2B_1}$ that are dependent on the others.

Theorem 4.6 *If $f : U \longrightarrow V$ is a linear map between two finite vector spaces then:*

$$\boxed{dim(ker(f)) + dim(im(f)) = dim(U)}$$

Proof: Suppose that $dim(ker(f)) = k$ and $dim(U) = m$. Let

$$B_k = \{\bar{u}_1, \ldots, \bar{u}_k\}$$

be a basis of the kernel of f. We can complete this base up to a basis of U:

$$B_1 = \{\bar{u}_1, \ldots, \bar{u}_k, \bar{u}_{k+1}, \ldots, \bar{u}_m\}$$

We know that $im(f)$ is spanned by $f(B_1)$:

$$Im(f) = \mathcal{L}\{f(\bar{u}_1), \dots, f(\bar{u}_k), f(\bar{u}_{k+1}), \dots, f(\bar{u}_m)\} = \mathcal{L}\{f(\bar{u}_{k+1}), \dots, f(\bar{u}_m)\}$$

since by definition of the kernel $f(\bar{u}_1) = \dots = f(\bar{u}_k) = \bar{0}$.

So $\{f(\bar{u}_{k+1}), \dots, f(\bar{u}_m)\}$ is a spanning set of $im(f)$. Let us see that is an independent set. Suppose that

$$\alpha^{k+1} f(\bar{u}_{k+1}) + \dots + \alpha^m f(\bar{u}_m) = \bar{0}.$$

Then

$$f(\alpha^{k+1}\bar{u}_{k+1} + \dots + \alpha^m \bar{u}_m) = \bar{0} \quad \Rightarrow \quad \alpha^{k+1}\bar{u}_{k+1} + \dots + \alpha^m \bar{u}_m \in ker(f).$$

We next use that B is a basis of $ker(f)$:

$$\alpha^{k+1}\bar{u}_{k+1} + \dots + \alpha^m \bar{u}_m = \alpha^1 \bar{u}_1 + \dots + \alpha^k \bar{u}_k$$

and bringing all terms to the first member of the equation

$$\alpha^1 \bar{u}_1 + \dots + \alpha^k \bar{u}_k - \alpha^{k+1}\bar{u}_{k+1} - \dots - \alpha^m \bar{u}_m = \bar{0}$$

Since the vectors in B' are a basis, in particular they are independent. We deduce that $\alpha^{k+1} = \dots = \alpha^m = 0$ and we have proved the required linear independence.

In summary, we have proved that $\{f(\bar{u}_{k+1}), \dots, f(\bar{u}_m)\}$ is an independent, spanning set of $im(f)$, that is, a basis. So

$$dim(im(f)) = m - (k+1) + 1 = m - k = dim(U) - dim(ker(f)).$$

5 The matrix associated to a projection mapping.

We have seen that given two complementary vector subspaces S_1, S_2, every vector $\bar{x} \in V$ can be uniquely expressed as $\bar{x} = \bar{x}_1 + \bar{x}_2$, with $\bar{x}_1\ inS_1$ and $\bar{x}_2 \in S_2$.

This allows us to define the projection map onto S_1 along S_2:

$$p_1 : V \longrightarrow V, \qquad p_1(\bar{x}) = \bar{x}_1 \text{ if } \bar{x} = \bar{x}_1 + \bar{x}_2 \text{ with } \bar{x}_1 \in S_1 \text{ and } \bar{x}_2 \in S_2$$

The steps to find the associated matrix of a projection mapping with respect to a basis C (which normally will be the canonical basis) are:

1. Find bases $\{u_1, u_2, \dots, u_k\}$ and $\{v_1, v_2 \dots, v_l\}$ respectively of the subspaces S_1 and S_2, expressing their vectors in coordinates with respect to the basis C.
2. Form a base of the space V by joining the previous bases (the fact that the subspaces are complementary guarantees that the union of the basis is a base of the total space):
$$B = \{\underbrace{u_1, u_2, \dots, u_k}_{S_1}, \underbrace{v_1, v_2 \dots, v_l}_{S_2}\}.$$
3. With respect to the basis B above, the associated matrix is a diagonal matrix, with as many ones on the diagonal as the dimension k of S_1 and as many zeros as the dimension l of S_2 .
$$P_B = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underbrace{0 \quad 0 \quad \dots \quad 0}_{dim(S_1)=k} & & & & \underbrace{0 \quad 0 \quad \dots \quad 0}_{dim(S_2)=l} \end{pmatrix}.$$
The matrix P_B has this form because the vectors of S_1, when projected onto S_1, remain invariant. On the contrary, the vectors S_2, when projected along to that same subspace, go to the zero vector.
4. Finally we make a change of basis to express the associated matrix in the starting base C:
$$P_C = M_{CB} P_B M_{BC} = M_{CB} P_M M_{CB}^{-1}.$$

6 Composition of linear maps.

Proposition 6.1 *Let $f : U \longrightarrow V$ and $g : V \longrightarrow W$ be two linear maps. Then the composition map $g \circ f$ is also a linear map.*

Proof: Let $\bar{x}, \bar{x}' \in U$ and $\alpha, \beta \in \mathbb{K}$. We have:

$$\begin{aligned}(g \circ f)(\alpha\bar{x} + \beta\bar{x}') &= g(f(\alpha\bar{x} + \beta\bar{x}')) = \qquad \text{(linearity of } f) \\ &= g(\alpha f(\bar{x}) + \beta f(\bar{x}')) = \qquad \text{(linearity of } g) \\ &= \alpha g(f(\bar{x})) + \beta g(f(\bar{x}')) = \alpha(g \circ f)(\bar{x}) + \beta(g \circ f)(\bar{x}').\end{aligned}$$

■

If we have bases in each of the vector spaces U, V and W:

$$B_1 = \{\bar{u}_1, \dots, \bar{u}_m\}, \qquad B_2 = \{\bar{v}_1, \dots, \bar{v}_n\}, \qquad B_3 = \{\bar{w}_1, \dots, \bar{w}_p\}.$$

and we call $h = g \circ f$ the composition map, we can see how the matrices $F_{B_2B_1}$, $G_{B_3B_2}$ and $H_{B_3B_1}$ are related. If we denote the coordinates of the vectors $\bar{x}$, $\bar{y} = f(\bar{x})$, $\bar{z} = h(\bar{x}) = g(\bar{y})$, by (x), (y), (z) with respect to the bases B_1, B_2 and B_3 respectively, we know that

$$(y) = F_{B_2B_1}(x); \qquad (z) = G_{B_3B_2}(y); \qquad (z) = H_{B_3B_1}(x)$$

Therefore:

$$(z) = G_{B_3B_2}(y) = G_{B_3B_2}F_{B_2B_1}(x)$$

and we deduce that:

$$\boxed{\text{If } h = g \circ f, \qquad H_{B_3B_1} = G_{B_3B_2}F_{B_2B_1}.}$$

7 Classification of linear maps.

7.1 Monomorphisms.

Definition 7.1 *A* **monomoprhism** *is an injective linear map.*

We next collect some properties of monomorphisms.

1. *A linear map $f : U \longrightarrow V$ es injective $\iff$ $ker(f) = \{\bar{0}\}$.*

   Proof:

   $\Longrightarrow$: Suppose that $f : U \longrightarrow W$ is an injective linear map. Then:

   $$\bar{x} \in ker(f) \;\Rightarrow\; f(\bar{x}) = \bar{0} \;\Rightarrow\; f(\bar{x}) = f(\bar{0}) \;\underset{f \text{ injective}}{\Rightarrow}\; \bar{x} = \bar{0}.$$

   and thus $ker(f) = \{\bar{0}\}$.

   $\Longleftarrow$: Suppose that $ker(f) = \{\bar{0}\}$. Then:

   $$\begin{array}{lllllll} f(\bar{x}) = f(\bar{x}') & \Rightarrow & f(\bar{x}) - f(\bar{x}') = \bar{0} & \Rightarrow & f(\bar{x} - \bar{x}') = \bar{0} & \Rightarrow & \\ & \Rightarrow & \bar{x} - \bar{x}' \in ker(f) & \Rightarrow & \bar{x} - \bar{x}' = \bar{0} & \Rightarrow & \bar{x} = \bar{x}' \end{array}$$

   and therefore f is injective.

2. *If $f : U \longrightarrow V$ is a monomorphism between vector spaces of finite dimension then $dim(U) \leq dim(V)$.*

   Proof: It is sufficient to note that $im(f) \subset V$ and also that (because f is injective) $ker(f) = \{\bar{0}\}$. Then:

   $$dim(U) = dim(im(f)) + dim(ker(f)) = dim(im(f)) \leq dim(V).$$

7.2 Epimorphisms.

Definition 7.2 *An* **epimorphism** *is a surjective linear map.*

The following are some properties of epimorphisms:

1. *If $f : U \longrightarrow V$ is an epimorphism between vector spaces of finite dimension then $dim(U) \geq dim(V)$.*

   Proof: It is sufficient to note that $im(f) = V$, because f is surjective. Then:

   $$dim(U) = dim(im(f)) + dim(ker(f)) \geq dim(im(f)) = dim(V).$$

7.3 Isomorphisms.

Definition 7.3 *An* **isomorphism** *is a bijective linear map.*

The following are some properties of isomorphisms:

1. *If f is a linear map, [f is bijective $\iff$ $ker(f) = \bar{0}$ and $im(f) = U$].*

   Proof: It is sufficient to note that:

   - f bijective $\iff$ f injective and surjective $\iff$ $ker(f) = \bar{0}$ and $im(f) = U$.

2. *If $f : U \longrightarrow V$ is an isomorphism, then its inverse map $f^{-1} : V \longrightarrow U$ is an isomorphism.*

   Proof: Since f is bijective, we know that its inverse map f^{-1} exists and is also bijective. It remains to prove that f^{-1} is linear.

   Let $\bar{y}, \bar{y}' \in V$ and $\alpha, \beta \in \mathbb{K}$. Suppose that $f^{-1}(\bar{y}) = \bar{x}$ and $f^{-1}(\bar{y}') = \bar{x}'$. Then:

   $$\left.\begin{array}{rcl} f^{-1}(\bar{y}) = \bar{x} & \Rightarrow & f(\bar{x}) = \bar{y} \\ f^{-1}(\bar{y}') = \bar{x}' & \Rightarrow & f(\bar{x}') = \bar{y}' \end{array}\right\} \Rightarrow f(\alpha\bar{x} + \beta\bar{x}') = \alpha f(\bar{x}) + \beta f(\bar{x}') = \alpha\bar{y} + \beta\bar{y}'.$$

   Therefore:

   $$f^{-1}(\alpha\bar{y} + \beta\bar{y}') = \alpha\bar{x} + \beta\bar{x}' = \alpha f^{-1}(\bar{y}) + \beta f^{-1}(\bar{y}').$$

3. *The composition of two isomorphisms is an isomorphism.*

   It is sufficient to note that the composition of linear maps is a linear map and that the composition of bijective maps is a bijective map.

4. *Any two spaces with the same dimension are isomorphic.*

   Proof: We can prove it in two ways:

   - We have seen that any vector space of dimension n is isomorphic to $\mathbb{K}^n$. Thus two spaces of the same dimension are isomorphic to each other.

   - Directly, if U and V are two n-dimensional vector spaces and we have bases

   $$B_1 = \{\bar{u}_1, \ldots, u_n\}; \qquad B_2 = \{\bar{v}_1, \ldots, \bar{v}_n\}$$

respectively of U and V, we can define the linear map $f : U \longrightarrow V$ as the one which acts on the base B_1 as follows:

$$f(\bar{u}_1) = \bar{v}_1; \quad \dots \quad f(\bar{u}_n) = \bar{v}_n;$$

The matrix associated to f with respect to the bases B_1 and B_2 is $F_{B_2B_1} = Id$. Therefore this map is invertible. Its associated matrix is $F'_{B_1B_2} = (F_{B_2B_1})^{-1} = Id$. Any map which has an inverse is bijective.

5. *If there is an isomorphism $f : U \longrightarrow V$ between U and V then $dim(U) = dim(V)$.*

   Proof: Since f is injective we know that $dim(U) \leq dim(V)$. Becuase f is surjective, $dim(U) \geq dim(V)$. Combining both facts we obtain the equality of the dimensions.

7.4 Endomorphisms.

Definition 7.4 *An* **endomorphism** *is a linear map from a vector space to itself:*

$$f : U \longrightarrow U.$$

Definition 7.5 *An* **automorphism** *is a bijective endomorphism.*

In the next chapter we will study endomorphisms of finite vector spaces in detail.

8 Vector space of homomorphisms.

Given two vector spaces U, V we denote by $Hom(U, V)$ the set of all linear maps from U to V. On this set one can define two operations: the sum of maps (internal) and the product by a scalar (external):

- The sum of two linear maps is defined as

$$(f + g)(\bar{x}) = f(\bar{x}) + g(\bar{x}), \qquad \forall f, g \in Hom(U, V), \quad \forall \bar{x} \in U.$$

- The product of a linear mapping by a scalar is defined as

$$(\lambda f)(\bar{x}) = \lambda f(\bar{x}), \qquad \forall f \in Hom(U, V), \quad \lambda \in \mathbb{K}, \quad \forall \bar{x} \in U.$$

It is easy to see that with these operations $Hom(U, V)$ has a vector space structure over $\mathbb{K}$.

Proposition 8.1 *If U, V are vector spaces of dimension m and n respectively, $Hom(U, V)$ is a vector space of dimension $n \cdot m$.*

Proof: Choose two bases B_1 and B_2 of U and V respectively.

$$B_1 = \{\bar{u}_1, \dots, \bar{u}_m\}; \qquad B_2 = \{\bar{v}_1, \dots, \bar{v}_2\}.$$

We define the following map between the vector spaces $Hom(U, V)$ and $\mathcal{M}_{n\times m}(\mathbb{K})$:

$$\begin{array}{rccc} \pi : & Hom(V, U) & \longrightarrow & \mathcal{M}_{n\times m}(\mathbb{K}) \\ & f & \longrightarrow & F_{B_2B_1} \end{array}$$

which associates to each linear map in $Hom(U, V)$ its associated matrix with respect to the previously fixed bases.

The map π satifies:

- *It is linear.* Indeed,

$$\pi(\alpha f + \beta g) = \alpha F_{B_2B_1} + \beta G_{B_2B_1} = \alpha\pi(f) + \beta\pi(g)$$

for any $\alpha, \beta \in \mathbb{K}$ and $f, g \in Hom(U, V)$.

- *It is injective.* Indeed,

$$f \in ker(\pi) \quad \Rightarrow \quad F_{B_2B_1} = \Omega \quad \Rightarrow \quad f = 0.$$

- *It is surjective.* Indeed, given any matrix $F \in \mathcal{M}_{n\times m}(\mathbb{K})$ we can always define a linear map of U on V whose associated matrix with respect to B_1 and B_2 is F. Just take

$$f(x^j) = (\bar{v}_i)F(x^j),$$

where (x^j) are the coordinates of any vector of U with respect to the basis B_1.

Thus π is an isomorphism and

$$dim(Hom(V, U)) = dim(\mathcal{M}_{n\times m}(\mathbb{K}) = n \cdot m.$$

8.1 Dual space.

As a particular case of a vector space of homomorphisms, we define

Definition 8.2 *Given a vector space U over the field $\mathbb{K}$, the* **dual space of** *U, denoted by U^*, is the vector space $Hom(U, \mathbb{K})$.*

The elements of U^* are linear maps:

$$f : U \longrightarrow \mathbb{K}$$

and are often called **linear forms** *or* **covectors.**

It is clear from the previous discussion that $dim(U^*) = dim(U)$.