2. Spanning sets. Linear independence. Bases.

1 Linear combinations of vectors.

1.1 Linear combination.

Definition 1.1 *Let V be a vector space over $\mathbb{K}$. A* **linear combination** *of a set of vectors $\{\bar{x}_1, \dots, \bar{x}_n\}$ is any vector of the form:*

$$\bar{y} = \alpha^1 \bar{x}_1 + \dots + \alpha^n \bar{x}_n, \quad \text{with} \quad \alpha^1, \dots, \alpha^n \in \mathbb{K}.$$

1.2 Linear span.

Definition 1.2 *Let V be a vector space over $\mathbb{K}$. Given a subset $A \subset V$ the* **linear span of** *A is the set of vectors that are linear combinations of elements of A:*

$$\mathcal{L}\{A\} = \langle A \rangle = \{\sum \alpha^i \bar{x}_i | \, \alpha^i \in \mathbb{K}, \, \bar{x}_i \in A\}$$

The set A is called **generating set or spanning set** *of $\mathcal{L}\{A\}$. If $\mathcal{L}\{A\} = V$ the set A is called* **spanning set or generating set** *of the vector space.*

Let us see an equivalent definition of linear span:

Proposition 1.3 *Let V be a vector space over $\mathbb{K}$ and A a subset V. The linear span of A is the smallest linear subspace that contains the set A.*

Proof: First, let us see $\mathcal{L}\{A\}$ is a vector subspace. It is sufficient to note that $\bar{0} \in \mathcal{L}\{A\}$ and if $\bar{x}, \bar{y}$ are linear combinatios of elements of A, then $\alpha\bar{x} + \beta\bar{y}$ is a linear combination of elements of A for any $\alpha, \beta \in \mathbb{K}$.

On the other hand, it is clear that $A \subset \mathcal{L}\{A\}$.

Finally, let us see that any subspace $S \subset A$ satisfies $\mathcal{L}\{A\} \subset S$. If $\bar{v} \in \mathcal{L}\{A\}$:

$$\bar{v} = \lambda^1 \bar{a}_1 + \dots + \lambda^n \bar{a}_n \quad \text{with} \quad \bar{a}_i \in A \subset S.$$

Since S is a vector subspace, all linear combinations of elements of S remain in S. Therefore, $\bar{v} \in S$. ■

From the definition and this proposition we deduce some properties:

1. $A \subset B \quad \Rightarrow \quad \mathcal{L}\{A\} \subset \mathcal{L}\{B\}$.
2. $\bar{x} \in \mathcal{L}\{A\} \quad \Rightarrow \quad \mathcal{L}\{A \cup \{\bar{x}\}\} = \mathcal{L}\{A\}$.
   Proof: Note that
   $$A \subset A \cup \{\bar{x}\} \quad \Rightarrow \quad \mathcal{L}\{A\} \subset \mathcal{L}\{A \cup \{\bar{x}\}\},$$
   and $\mathcal{L}\{A\}$ is a vector subspace containing $A \cup \{\bar{x}\}$.
3. $\mathcal{L}\{\mathcal{L}\{A\}\} = \mathcal{L}\{A\}$.
   Proof: This follows from the fact that $\mathcal{L}\{A\}$ is the smallest subspace containing $\mathcal{L}\{A\}$.
4. *Any vector subspace is the linear span of some set.*
   Proof: Note that S is the smallest subspace containing S, so $\mathcal{L}\{S\} = S$.

1.3 Equivalent sets of vectors.

Definition 1.4 *Let V be a vector space over $\mathbb{K}$. Two subsets A, B of V are called* **equivalent** *if they span the same linear subspace:*

$$A, B \text{ equivalent} \iff \mathcal{L}\{A\} = \mathcal{L}\{B\}.$$

Proposition 1.5 *Two sets A and B are equivalent if and only if $A \subset \mathcal{L}\{B\}$ and $B \subset \mathcal{L}\{A\}$.*

Proof:

$\Longrightarrow$:

If A, B are equivalent, $\mathcal{L}\{A\} = \mathcal{L}\{B\}$ and from this $A \subset \mathcal{L}\{B\}$ and $B \subset \mathcal{L}\{A\}$.

$\Longleftarrow$:

Suppose that $A \subset \mathcal{L}\{B\}$ and $B \subset \mathcal{L}\{A\}$. Then:

$$\begin{array}{lcl} A \subset \mathcal{L}\{B\} & \Rightarrow & \mathcal{L}\{A\} \subset \mathcal{L}\{\mathcal{L}\{B\}\} = \mathcal{L}\{B\}. \\ B \subset \mathcal{L}\{A\} & \Rightarrow & \mathcal{L}\{B\} \subset \mathcal{L}\{\mathcal{L}\{A\}\} = \mathcal{L}\{A\}. \end{array}$$

2 Linear dependence and linear independence of vectors.

Definition 2.1 *A set of vectors $\{\bar{x}_1, \dots, \bar{x}_n\}$ is said to be* **linearly independent** *if:*

$$\alpha^1 \bar{x}_1 + \dots + \alpha^n \bar{x}_n = \bar{0} \quad \Rightarrow \quad \alpha^1 = \dots = \alpha^n = 0.$$

In other case we will say that they are **linearly dependent**.

Proposition 2.2 *A set of vectors $\{\bar{x}_1, \dots, \bar{x}_n\}$ is linearly dependent if and only if one of the vectors is a linear combination of the others.*

Proof:

$\Longrightarrow$:

Suppose they are linearly dependent. Then there is a linear combination:

$$\alpha^1\bar{x}_1 + \ldots + \alpha^n\bar{x}_n = \bar{0}$$

with some $\alpha^{i_0} \neq 0$. This expression can be written as:

$$\bar{x}_{i_0} = \underbrace{\frac{\alpha^1}{\alpha^{i_0}}\bar{x}_1 + \ldots + \frac{\alpha^n}{\alpha^{i_0}}\bar{x}_n}_{\text{term } i_0 \text{ does not appear}}$$

and we see that $\bar{x}_{i_0}$ is a linear combination of the others.

$\Longleftarrow$. Suppose that one of the vectors in the set, say $\bar{x}_{i_0}$, is a linear combination of the others. Then:

$$\bar{x}_{i_0} = \underbrace{\alpha^1\bar{x}_1 + \ldots + \alpha^n\bar{x}_n}_{\text{term } i_0 \text{ does not appear}} \quad \Rightarrow \quad \alpha^1\bar{x}_1 + \ldots + \alpha^{i_0}\bar{x}_{i_0} + \ldots + \bar{\alpha}^n\bar{x}_n = \bar{0}$$

with $\alpha^{i_0} = -1 \neq 0$ and hence $\{\bar{x}_1, \ldots, \bar{x}_n\}$ are linearly dependent. ■

Remark 2.3 *From the definition and this proposition we deduce some properties::*

1. *Any set of vectors that contains the vector $\bar{0}$ is dependent.*
2. *The empty set is considered independent.*
3. *If we remove a vector from a linearly independent set, the set we obtain is independent.*
4. *If we add a vector $\bar{x}$ to a linearly independent set, such that $\bar{x}$ is not a linear combination of the vectors in the set, then the new set is independent.*

   Proof: Suppose $\{\bar{x}_1, \ldots, \bar{x}_n\}$ are independent; we add the vector $\bar{x}$. If the new set were linearly dependent

   $$\lambda\bar{x} + \lambda^1\bar{x}_1 + \ldots + \lambda^n\bar{x}_n = 0$$

   with some λ or λ^i not null. Since the initial set is independent, necessarily $\lambda \neq 0$. We obtain:

   $$\bar{x} = -\frac{\lambda^1}{\lambda}\bar{x}_1 - \ldots - \frac{\lambda^n}{\lambda}\bar{x}_n$$

   and therefore $\bar{x}$ would be a linear combination of the initial set.
5. *If we add a vector to a dependent set, the new set is still dependent.*
6. **If A is a infinite set, we say that is independent when all its finite subsets are independent.**

3 Bases, dimension and coordinates.

3.1 Bases and dimension.

Definition 3.1 *Given a vector space V, a ordered set of vectors $B = \{\bar{x}_1, \ldots, \bar{x}_n\}$ is said to be a* **basis** *when it is an independent set and it spans V.*

Remark 3.2 *One can define bases with a infinite number of vectors. However, through this course we will only work with finite bases.*

Proposition 3.3 *Let V be a vector space. If $A = \{\bar{x}_1, \ldots, \bar{x}_n\}$ is a spanning set of V, then A contains a basis of V.*

Proof: If A is an independent set we are done.

If not, one of the vectors in A, say $\bar{x}_{i_0}$, is a linear combination of the others. Then:

$$\bar{x}_{i_0} \in \mathcal{L}\{\bar{x}_1, \ldots, \bar{x}_{i_0-1}, \bar{x}_{i_o+1}, \ldots, \bar{x}_n\}$$

By the properties of the spanning set:

$$\mathcal{L}\{\bar{x}_1, \ldots, \bar{x}_{i_0-1}, \bar{x}_{i_o+1}, \ldots, \bar{x}_n\} = \mathcal{L}\{A\}$$

Now, the set $\{\bar{x}_1, \ldots, \bar{x}_{i_0-1}, \bar{x}_{i_o+1}, \ldots, \bar{x}_n\}$ is a spanning set of V. If A is independent we are done. In other case, we repeat the previous argument and we obtain a new spanning set with one less vector. Since there are a finite number of vectors in A, eventually the process will end and we will obtain a basis. ■

Remark 3.4 *This result is also true if A is an infinite set, but the proof is more delicate.*

Theorem 3.5 (Steinitz Theorem) *Let V be a vector space. Let $A = \{\bar{a}_1, \ldots, \bar{a}_m\}$ be an independent set and let $B = \{\bar{b}_1, \ldots, \bar{b}_n\}$ be a basis. Then $m \leq n$ and there are m vectors of B which can be replaced with those of A to obtain a new basis.*

Proof: If $A = \emptyset$ the result is trivial. Hence assume that $A \neq \emptyset$. Because A is an independent set, all the vectors of A are not zero. Since B is a basis:

$$\bar{a}_1 = \alpha^1\bar{b}_1 + \ldots + \alpha^n\bar{b}_n, \text{ with } \alpha^i \in \mathbb{K}, \text{ and some } \alpha^{i_0} \neq 0$$

For simplicity, we will suppose $i_0 = 1$. Then:

$$\bar{b}_1 = \frac{1}{\alpha^1}\bar{a}_1 - \frac{\alpha^2}{\alpha^1}\bar{b}_2 + \ldots + \frac{\alpha^n}{\alpha^1}\bar{b}_n$$

Applying Proposition 1.5, we see that the sets $\{\bar{b}_1, \ldots, \bar{b}_n\}$ and $\{\bar{a}_1, \bar{b}_2, \ldots, \bar{b}_n\}$ are equivalent and therefore

$$\mathcal{L}\{\bar{a}_1, \bar{b}_2, \ldots, \bar{b}_n\} = \mathcal{L}\{\bar{b}_1, \ldots, \bar{b}_n\} = V$$

Moreover, $\bar{a}_1$ cannot be a linear combination of $\bar{b}_2, \ldots, \bar{b}_n$ because in that case $\bar{b}_1$ would be too, and B would not be independent. We deduce that $\{\bar{a}_1, \bar{b}_2, \ldots, \bar{b}_n\}$ is independent, and hence a basis of V.

If $m = 1$ we are done. In other case, we repeat the process. Now we have:

$$\bar{a}_2 = \beta^1 \bar{a}_1 + \beta^2 \bar{b}_2 \ldots + \alpha^n \bar{b}_n, \text{ with } \beta^i \in \mathbb{K}, \text{ and some } \beta^{i_0} \neq 0$$

with $i_0 > 1$ because A is independent. We suppose that $\beta^2 \neq 0$ and repeat the previous argument.

We will obtain that $\{\bar{a}_1, \bar{a}_2, \bar{b}_3, \ldots, \bar{b}_n\}$ is a basis of V.

We repeat this process until we finish with the m vectors of A. We note that it cannot happen that $m > n$, since in the n-th step we would obtain that A is a basis of V, and therefore a_{n+1} would be a linear combination of other elements of A, thus contradicting the hypothesis that they are independent. ■

This Theorem has important consequences:

1. **All bases of a vector space with a finite spanning set have the same number of elements.**
   Proof: Given two finite bases B and B', with m and n vectors respectively, we can apply Steinitz's Theorem in two ways. Taking B as the independent set and B' as the basis, we obtain $m \leq n$; taking B' as the independent set and B as the basis, we obtain $n \leq m$. We deduce that $n = m$.
2. A vector space with a finite spanning set is said to be finite dimensional:

   The number of elements of a basis of a vector space is called the dimension of the vector space.
3. A vector space without any finite spanning set is said to have infinite dimension.
4. Any linearly independent set A can be completed to a basis, by choosing suitable vectors from any basis B.
5. **Any linearly independent set with n vectors in a vector space of dimension n is actually a basis.**
6. **Any spanning set with n vectors in a vector space of dimension n is a basis.**

3.2 Coordinates with respect to a basis.

Definition 3.6 *Let V be a finite dimensional vector space and $B = \{\bar{e}_1, \ldots, \bar{e}_n\}$ a basis of V. Given any vector $\bar{x} \in V$ we call* **coordinates of $\bar{x}$ with respect to the basis** B *the scalars $(x^1, \ldots, x^n)$ satisfying*

$$\bar{x} = x^1 \bar{e}_1 + \ldots + x^n \bar{e}_n.$$

Let us see that this definition makes sense.

Fist of all, since B is a basis, it is a spanning set and thus any vector $\vec{x} \in V$ can be written as a linear combination of the elements of B. Therefore, the scalars $(x^1, \ldots, x^n)$ satisfying the required condition exist.

On the other hand, let us see that the **coordinates of a vector with respect to a basis are unique**. Suppose that both $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ satisfy the definition, that is:

$$x^1 \bar{e}_1 + \ldots + x^n \bar{e}_n = \bar{x} = y^1 \bar{e}_1 + \ldots + y^n \bar{e}_n \quad \Rightarrow \quad (x^1 - y^1)\bar{e}_1 + \ldots + (x^n - y^n)\bar{e}_n = 0.$$

Because B is a basis, its vectors are linearly independent and we deduce that:

$$x^1 = y^1, \quad \ldots, \quad x^n = y^n \qquad \Rightarrow \quad (x^1, \ldots, x^n) = (y^1, \ldots, y^n).$$

It is important to note that the coordinates of a vector depend on the order of the vectors in the basis.

3.3 Isomorphism between $\mathbb{K}^n$ and any vector space of dimension n.

Given any vector space V over $\mathbb{K}$ and a basis $B = \{\bar{e}_1, \ldots, \bar{e}_n\}$ of V, we can define the following bijective function which takes each vector to its coordinates:

$$\begin{array}{ccll} V & \longrightarrow & \mathbb{K}^n & \\ \bar{x} & \longrightarrow & (x^1, x^2, \ldots, x^n) & \quad \text{where } \bar{x} = x^1 \bar{e}_1 + \ldots + x^n \bar{e}_n. \end{array}$$

It is well defined because, as we have seen, every vector of V is uniquely expressed as a linear combination of the elements of B.

It is injective, because

$$(x^1, x^2, \ldots, x^n) = (y^1, y^2, \ldots, y^n) \quad \Rightarrow \quad x^1 \bar{e}_1 + \ldots + x^n \bar{e}_n = y^1 \bar{e}_1 + \ldots + y^n \bar{e}_n.$$

It is surjective, because given $(x^1, x^2, \ldots, x^n) \in \mathbb{K}^n$ there exists $\bar{x} \in V$ with:

$$\bar{x} = x^1 \bar{e}_1 + \ldots + x^n \bar{e}_n.$$

Moreover this function preserves the operations of V and $\mathbb{K}^n$, that is:

- coordinates of $(\bar{x} + \bar{y})$ =coordinates of $\bar{x}$+coordinates of $\bar{y}$.

- coordinates of $\lambda \bar{x} = \lambda\cdot$ coordinates of $\bar{x}$.

This means that **working on a vector space V of dimension n is equivalent to working on the vector space $\mathbb{K}^n$** provided we have fixed a basis of V.

3.4 Canonical bases.

Intuitively, **canonical bases** are the simplest bases of certain vector spaces. Let us see the most common of them:

1. In $\mathbb{K}^n$ the canonical basis is
$$\{(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,1)\}.$$
These vectors are usually denoted by $\{\bar{e}_1,\bar{e}_2,\ldots,\bar{e}_n\}$.
2. In the vector space of polynomials of degree at most n with coefficients in $\mathbb{K}$, $\mathcal{P}_n(\mathbb{K})$, the canonical basis is $\{1,x,x^2,\ldots,x^n\}$.
3. In the vector space of matrices $\mathcal{M}_{m\times n}(\mathbb{K})$ the canonical basis is formed by all the matrices that have one entry equal to 1 and all the others are 0s. They are usually ordered first by row index and then by column index of the non-null entry. For example in $\mathcal{M}_{2\times 3}(\mathbb{K})$ the canonical base is:
$$\left\{\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}\right\}.$$
4. In the vector space of symmetric matrices $\mathcal{S}_{n\times n}(\mathbb{K})$ the canonical basis is formed by all the matrices that have an 1 somewhere on the diagonal and the remaining entries are zeros, or two elements equal to 1 on symmetrical positions with respect to the diagonal and zeros on the remaining ones. For example in $\mathcal{S}_{3\times 3}(\mathbb{K})$ the canonical basis is:
$$\left\{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\}$$

4 Rank of a set of vectors.

Definition 4.1 *Let V be a vector space and $A=\{\bar{x}_1,\ldots,\bar{x}_m\}$ a set of vectors of V. The* **rank of** *A is the dimension of the subspace spanned by A or, equivalently the number of independent vectors of A.*

Theorem 4.2 *Let V be a vector space, $A=\{\bar{x}_1,\ldots,\bar{x}_m\}$ a set of vectors of V and $B=\{\bar{e}_1,\ldots,\bar{e}_n\}$ a basis of V. Suppose that the coordinates of the vector $\bar{x}_i$ are $(\lambda_i^1,\ldots,\lambda_i^n)$. We consider the matrix M formed by these coordinates:*
$$M=\begin{array}{c}\begin{pmatrix}\lambda_1^1&\lambda_2^1&\ldots&\lambda_m^1\\\lambda_1^2&\lambda_2^2&\ldots&\lambda_m^2\\\vdots&\vdots&\ddots&\vdots\\\lambda_1^n&\lambda_2^n&\ldots&\lambda_m^n\end{pmatrix}\\\begin{matrix}\uparrow&\uparrow&&\uparrow\\\bar{x}_1&\bar{x}_2&\ldots&\bar{x}_m\end{matrix}\end{array}$$
Then
$$rank(A)=rank(M).$$

Proof: Recall that the rank of M is the highest order of a nonzero minor of A. Also, the rank of a matrix does not change if we perform elementary column operations on it. On the other hand, due to the properties of the linear span, the space spanned by any given set of vectors does not change if we replace a vector of A by the same vector added to a linear combination of the remaining ones, or if we multiply a vector of A by a scalar, or if we change the order of the vectors in A.

In this way we can do elementary column operations on M until we get a reduced form M'. The columns of M' correspond to the coordinates with respect to the base B of m vectors $A'=\{\bar{z}_1,\ldots,\bar{z}_m\}$ which form a set equivalent to A. So:
$$rank(A)=rank(A'),\qquad rank(M)=rank(M').$$

We know that the rank of M' is the number of nonzero columns in the matrix; on the other hand, the non-null vectors of A' are linearly independent, in other case one of them would be a linear combination of the others and we could do column operations on M' resulting in one more null row. But M' is already the reduced form. From this:
$$\begin{array}{rcl}rank(A')&=&\text{number of non-zero vectors in } A'\\&=&\text{number of non-zero columns of } M' \quad = \quad rank(M').\end{array}$$
■

Corollary 4.3 *The determinant of a square matrix T is null if and only if one of its columns is a linear combination of the others.*

Proof: Fix $T\in\mathcal{M}_{n\times n}(\mathbb{K})$. Its columns can be considered as the coordinates of n vectors $A=\{\bar{x}_1,\ldots,\bar{x}_n\}$ with respect to the canonical base of $\mathbb{K}^n$.

Suppose $det(T)=0$. Then $rank(T)<n$ and therefore $rank(A)<n$. This means that the n vectors of A are not a basis and therefore one of them is a linear combination of the others. Therefore, the corresponding column of coordinates will be a linear combination of the other columns.

Conversely if a column is a linear combination of the others $rank(A)<n$ and therefore $rank(T)<n$. But then there cannot be a minor of order n with non-zero determinant and therefore $det(T)=0$. ■

5 Change of basis.

Suppose we have two bases in a vector space V :
$$B=\{\bar{e}_1,\ldots,\bar{e}_n\};\quad B'=\{\bar{e}'_1,\ldots,\bar{e}'_n\}.$$

We are looking for a formula that relates the coordinates of a vector with respect to each one of these bases.

Since B is a basis, the vectors of B' can be expressed in coordinates with respect to the basis B:

$$\left.\begin{array}{l} \bar{e}'_1 = c_1^1\bar{e}_1 + c_1^2\bar{e}_2 + \ldots + c_1^n\bar{e}_n \\ \bar{e}'_2 = c_2^1\bar{e}_1 + c_2^2\bar{e}_2 + \ldots + c_2^n\bar{e}_n \\ \qquad\vdots \\ \bar{e}'_n = c_n^1\bar{e}_1 + c_n^2\bar{e}_2 + \ldots + c_n^n\bar{e}_n \end{array}\right\} \iff$$

$$(\bar{e}'_1 \quad \bar{e}'_2 \quad \ldots \quad \bar{e}'_n) = (\bar{e}_1 \quad \bar{e}_2 \quad \ldots \quad \bar{e}_n)\begin{pmatrix} c_1^1 & c_2^1 & \ldots & c_n^1 \\ c_1^2 & c_2^2 & \ldots & c_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ c_1^n & c_2^n & \ldots & c_n^n \end{pmatrix}$$

We write the previous expression in a simplified form as:

$$(\bar{e}') = (\bar{e})M_{BB'}$$

where the matrix

$$M_{BB'} = \begin{array}{c} \begin{pmatrix} c_1^1 & c_2^1 & \ldots & c_n^1 \\ c_1^2 & c_2^2 & \ldots & c_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ c_1^n & c_2^n & \ldots & c_n^n \end{pmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & & \uparrow \\ \bar{e}'_1 & \bar{e}'_2 & \ldots & \bar{e}'_n \end{array} \end{array}$$

will be called **change-of-basis matrix or transition matrix from the basis B' to the basis B**. We note that **its columns are formed by the coordinates of the basis vectors B' with respect to the basis B**.

The **change-of-basis matrix is always nonsingular**. This is a direct consequence of the Corollary 4.3 from the previous section. If the matrix were singular, the vectors of B' would not be independent.

Suppose the coordinates of a vector $\bar{x}$ with respect to the bases B and B' are respectively:

$$(x_1, \ldots, x_n), \quad \text{and} \quad (x'_1, \ldots, x'_n).$$

Then:

$$x_1 \cdot \bar{e}_1 + \ldots + x_n \cdot \bar{e}_n = \bar{x} = x'_1 \cdot \bar{e}'_1 + \ldots + x'_n \cdot \bar{e}'_n.$$

With matrix notation

$$(\bar{e}_1 \quad \bar{e}_2 \quad \ldots \quad \bar{e}_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (\bar{e}'_1 \quad \bar{e}'_2 \quad \ldots \quad \bar{e}'_n)\begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} \quad \text{or} \quad (\bar{e})(x) = (\bar{e}')(x').$$

If we now apply the expression of the change of basis, we get

$$(\bar{e})(x) = (\bar{e})M_{BB'}(x'),$$

and from the uniqueness of the coordinates with respect to the basis B we obtain:

$$\boxed{(x) = M_{BB'}(x')}$$

Note the following facts about the notation we have used:

1. The columns of the matrix $M_{BB'}$ are the coordinates of the vectors of the basis B' with respect to the basis B.
2. The matrix $M_{BB'}$ allows us to turn coordinates in the basis B' into coordinates in the basis B.
3. As a mnemonic rule to remember the base change formula, we note that the vector of column coordinates which appears multiplied by the change-of-basis matrix must be in the basis indicated by the subindex of M which is adjacent to it:
$$(x) = M_{B\boxed{B'}}\boxed{(x')}$$
4. The matrix $M_{B'B}$ is formed by the coordinates of the basis vectors B with respect to the basis B'.
5. The matrix $M_{B'B}$ will allow us to turn coordinates in the basis B into coordinates in the basis B':
$$\boxed{(x') = M_{B'B}(x)}$$
6. The following holds:
$$\boxed{M_{B'B} = (M_{BB'})^{-1}}$$
To check this, note that if we first change from basis B to basis B' and then undo the change by going back to the basis B, the coordinates thus obtained must be the initial ones, that is:
$$M_{BB'} \cdot M_{B'B} = Id \quad \Rightarrow \quad M_{BB'} = (M_{B'B})^{-1}.$$
7. If we have three bases C, B and B', we have the following relation between the change-of-basis matrices,
$$\boxed{M_{B'B} = M_{B'C} \cdot M_{CB}}$$
Note that again in this type of formulas the subscripts corresponding to the same bases are adjacent.
$$M_{B'B} = M_{B'\boxed{C}} \cdot M_{\boxed{C}B}$$

6 Equations of the subspaces.

Let V be a vector space and let S be a vector subspace of V. Suppose that $B = \{\bar{e}_1, \ldots, \bar{e}_n\}$ is a basis of V and that $\{\bar{s}_1, \ldots, \bar{s}_p\}$ is a basis of S. Let us see different ways to express the elements of S.

We will work with coordinates with respect to the basis B of V.

Given any vector $\bar{x} \in V$, we will denote its coordinates as $(x^1, \ldots, x^n)$ or simply (x^i).

The coordinates of the vectors $\bar{s}_i$ will be denoted by $(a_i^1, \ldots, a_i^n)$ or simply (a_i^j).

1. **Vector equation.**
   S is formed by the vectors $\bar{x}$ satisfying
   $$\boxed{\bar{x} = \alpha^1 \bar{s}_1 + \ldots + \alpha^p \bar{s}_p} \iff \boxed{\bar{x} = (\bar{s}_i)(\alpha^i)}$$
   for any $\alpha^i \in \mathbb{K}$.
2. **Parametric equations.**
   S is formed by the coordinate vectors $(x^1, \ldots, x^n)$ verifying:
   $$\boxed{\begin{array}{c} x^1 = a_1^1\alpha^1 + a_2^1\alpha^2 + \ldots + a_p^1\alpha^p \\ x^2 = a_1^2\alpha^1 + a_2^2\alpha^2 + \ldots + a_p^2\alpha^p \\ \vdots \\ x^n = a_1^n\alpha^1 + a_2^n\alpha^2 + \ldots + a_p^n\alpha^p \end{array}} \iff \boxed{x^j = \sum_i \alpha^i a_i^j, \qquad j = 1, 2, \ldots, n}$$
   for any $\alpha^i \in \mathbb{K}$. The α^i are the **parameters**. For a subspace S of dimension p, there must be p parameters.
3. **Implicit or Cartesian equations.**
   By eliminating parameters in the parametric equations we obtain $(n-p)$ independent implicit equations.
   S will be formed by the coordinate vectors $(x^1, \ldots, x^n)$ satisfying
   $$\boxed{\begin{array}{ccccccccc} b_1^1 x^1 & + & b_2^1 x^2 & + & \ldots & + & b_n^1 x^n & = 0 \\ b_1^2 x^1 & + & b_2^2 x^2 & + & \ldots & + & b_n^2 x^n & = 0 \\ & & & & \vdots & & & \\ b_1^{n-p} x^1 & + & b_2^{n-p} x^2 & + & \ldots & + & b_n^{n-p} x^n & = 0 \end{array}}$$

7 Dimension of the sum of subspaces.

Theorem 7.1 *Let V be a finite dimensional vector space. Let S_1 and S_2 be two subspaces of V. Then:*
$$\boxed{dim(S_1 + S_2) + dim(S_1 \cap S_2) = dim(S_1) + dim(S_2)}$$

Proof: Let $a = dim(S_1)$, $b = dim(S_2)$ and $r = dim(S_1 \cap S_2)$. Consider a basis of $S_1 \cap S_2$:
$$B = \{\bar{x}_1, \ldots, \bar{x}_r\}$$
Since $S_1 \cap S_2 \subset S_1$ and $S_1 \cap S_2 \subset S_1$, we know that we can complete up this basis both to one of S_1 and another one of S_2:
$$\begin{array}{ll} B_1 = \{\bar{x}_1, \ldots, \bar{x}_r, \bar{u}_{r+1}, \ldots, \bar{u}_a\} & \text{basis of } S_1. \\ B_2 = \{\bar{x}_1, \ldots, \bar{x}_r, \bar{v}_{r+1}, \ldots, \bar{v}_b\} & \text{basis of } S_2. \end{array}$$

Let us see that $B' = \{\bar{x}_1, \ldots, \bar{x}_r, \bar{u}_{r+1}, \ldots, \bar{u}_a, \bar{v}_{r+1}, \ldots, \bar{v}_b\}$ is a basis of $S_1 + S_2$:

- It is clear that it is a spanning set of $S_1 + S_2$ since:
$$\begin{aligned} S_1 + S_2 &= \mathcal{L}\{S_1 \cup S_2\} = \\ &= \mathcal{L}\{\bar{x}_1, \ldots, \bar{x}_r, \bar{u}_{r+1}, \ldots, \bar{u}_a, \bar{x}_1, \ldots, \bar{x}_r, \bar{v}_{r+1}, \ldots, \bar{v}_b\} = \\ &= \mathcal{L}\{\bar{x}_1, \ldots, \bar{x}_r, \bar{u}_{r+1}, \ldots, \bar{u}_a, \bar{v}_{r+1}, \ldots, \bar{v}_b\}. \end{aligned}$$

- Now let us see that they are linearly independent. Suppose we have an expression:
$$\lambda^1 \bar{x}_1 + \ldots + \lambda^r \bar{x}_r + \alpha^{r+1} \bar{u}_{r+1} + \ldots + \alpha^a \bar{u}_a + \beta^{r+1} \bar{v}_{r+1} + \ldots + \beta^b \bar{v}_b = \bar{0}$$

We want to see that then all the coefficients of this sum are zero.

The above expression can be written as:
$$\underbrace{\lambda^1 \bar{x}_1 + \ldots + \lambda^r \bar{x}_r + \alpha^{r+1} \bar{u}_{r+1} + \ldots + \alpha^a \bar{u}_a}_{\in S_1} = \underbrace{-\beta^{r+1} \bar{v}_{r+1} - \ldots - \beta^b \bar{v}_b}_{\in S_2} \qquad (*)$$

We deduce that both terms must be in $S_1 \cap S_2$, that is, they must be a linear combination of elements of B. In particular:
$$-\beta^{r+1} \bar{v}_{r+1} - \ldots - \beta^b \bar{v}_b = \gamma^1 \bar{x}_1 + \ldots + \gamma^r \bar{x}_r$$
or equivalently:
$$\gamma^1 \bar{x}_1 + \ldots + \gamma^r \bar{x}_r + \beta^{r+1} \bar{v}_{r+1} + \ldots + \beta^b \bar{v}_b = 0.$$
Since B_2 is a basis of S_2, it is an independent set and necessarily
$$\beta^{r+1} = \ldots = \beta^b = 0.$$

Now the first term of the equality $(*)$ is also $\bar{0}$ and since B_1 is a basis, its vectors are independent and necessarily
$$\lambda^1 = \ldots = \lambda^r = \alpha^{r+1} = \ldots = \alpha^a = 0.$$

■

As a consequence of this result and the reasoning we have done in the proof, we also conclude that:

Corollary 7.2 *Let U be a vector space of finite dimension. If S_1, S_2 are vector subspaces of U and $S_1 + S_2$ is a direct sum, then:*

$$dim(S_1 \oplus S_2) = dim(S_1) + dim(S_2).$$

Also if B_1 and B_2 are respectively bases of S_1 and of S_2, then $B_1 \cup B_2$ is a base of $S_1 \oplus S_2$.

By applying the same result to more than two vector subspaces, we obtain:

Corollary 7.3 *Let U be a vector space of finite dimension. If $S_1, \ldots, S_k$ are vector subspaces of U and $S_1 + \ldots + S_k$ is a direct sum, then:*

$$dim(S_1 \oplus \ldots \oplus S_k) = dim(S_1) + \ldots + dim(S_k).$$

Also if $B_1, \ldots, B_k$ are bases respectively of $S_1, \ldots, S_k$, then $B_1 \cup \ldots \cup B_k$ is a base of $S_1 \oplus \ldots \oplus S_k$.

8 Appendix: usual method of working on vector spaces.

An important feature of vector spaces is that once we have fixed a basis, all vectors in the space can be expressed by coordinates with respect to that basis. All the work we need to do on them can be reduced to operating with those coordinates through matrix calculations. This is independent of the nature of the specific vector space that we are handling (spaces R^n, function spaces, matrix spaces, etc...). We summarize this fact in the following scheme:

Method of working in a vector space:

1. **Fix a basis.**
2. **Express the data in coordinates with respect to the fixed basis.**
3. **Operate with these data (matrix calculation) until the desired result is obtained.**
4. **Express the result in the context of the starting vector space.**

8.1 Example:

We consider the following problem. Let U be the set of polynomials of degree less or equal to 2 that have 0 as a root and V those that have 1 as a root. Determine $U \cap V$, that is, the set of all polynomials of degree less or equal to 2 whose values at 0 and at 1 are both zero.

Evidently applying some basic properties of polynomials, one can quickly obtain that polynomials of degree less than or equal to 2 vanishing at 0 and 1 are exactly those which have the form

$$p(x) = kx(x-1) = kx^2 - kx.$$

However, we are going to solve the problem from the point of view of the theory of vector spaces.

We will work on the vector space $\mathcal{P}_2(R)$ of polynomials of degree less than or equal to 2. We follow the scheme described:

1. **Fix a basis.**
   The canonical basis of $\mathcal{P}_2(\mathbb{R})$ is $B = \{1, x, x^2\}$. We will work with respect to this basis. The coordinates of a polynomial $p(x) = a + bx + cx^2$ in this basis are (a, b, c).
2. **Express the data in coordinates with respect to the fixed basis..**
   The set U is:
   $$U = \{p(x) \in \mathcal{P}_2(\mathbb{R}) |\, p(0) = 0\}.$$
   If $p(x) = a + bx + cx^2$ then the condition defining the elements in U is:
   $$p(0) = 0 \quad \Leftrightarrow \quad a = 0.$$
   Therefore, in coordinates with respect to the base B, the subset is written as follows:
   $$U = \{(a, b, c) \in \mathcal{P}_2(\mathbb{R}) |\, a = 0\}.$$
   Since the condition $a = 0$ is linear, U is a vector subspace.
   Reasoning analogously for the set V:
   $$V = \{p(x) \in \mathcal{P}_2(\mathbb{R}) |\, p(1) = 0\}.$$
   In this way we obtain the implicit equation of V respect to the canonical basis:
   $$V = \{(a, b, c) \in \mathcal{P}_2(\mathbb{R}) |\, a + b + c = 0\}.$$
3. **Operate with these data (matrix calculation) until the desired result is obtained.**
   We need to calculate $U \cap V$. We have the implicit equations of both subspaces, so the intersection subspace is determined by these equations combined:
   $$U \cap V = \{(a, b, c) \in \mathcal{P}_2(\mathbb{R}) |\, a = 0, \quad a + b + c = 0\}.$$
   Since the two equations are independent and $dim(\mathcal{P}_2(\mathbb{R})) = 3$, the subspace $U \cap V$ has dimension $3 - 2 = 1$. We want to find a basis, that is, a single nonzero vector that satisfies both equations:
   $$a = 0, \quad a + b + c = 0 \qquad \Rightarrow \qquad a = 0, \quad b = -c.$$
   We can pick the vector $(0, 1, -1)$. Finally:
   $$U \cap V = \mathcal{L}\{(0, 1, -1)\}.$$

4. **Express the result in the context of the starting vector space.**
Now the polynomial with coordinates $(0, 1, -1)$ with respect to the base B is:

$$0 \cdot 1 + 1 \cdot x - 1 \cdot x^2 = x - x^2.$$

So

$$U \cap V = \mathcal{L}\{x - x^2\}.$$

In other words, the polynomials whose roots are 0 and 1 are exactly the multiples of the polynomial $x - x^2$.