Part III

Vector spaces.

1. Vector spaces and vector subspaces.

1 Vector spaces.

1.1 Definition.

Definition 1.1 *Let $\mathbb{K}$ be a conmutative field. A* **vector space over** $\mathbb{K}$ *is a non-empty set with two operations satisfying the following properties:*

1. *There is an internal operation $+ : V \times V \to V$ (called vector addition) such that $(V, +)$ is an abelian group, that is, it satisfies:*
   (a) *Associativity: $\bar{x} + (\bar{y} + \bar{z}) = (\bar{x} + \bar{y}) + \bar{z}$, for any $\bar{x}, \bar{y}, \bar{z} \in V$.*
   (b) *Identity element: $\exists \bar{0} \in V$ such that $\bar{0} + \bar{x} = \bar{x} + \bar{0} = \bar{x}$, for any $\bar{x} \in V$.*
   (c) *Inverse element: for any $\bar{x} \in V$, $\exists(-\bar{x}) \in V$ with*
   $$\bar{x} + (-\bar{x}) = (-\bar{x}) + \bar{x} = \bar{0}.$$
   (d) *Commutativity: $\bar{x} + \bar{y} = \bar{y} + \bar{x}$, for any $\bar{x}, \bar{y} \in V$.*
2. *There is an external operation $(\cdot) : \mathbb{K} \times V \longrightarrow V$ verifying:*
   (a) *$1 \cdot \bar{x} = \bar{x}$ for any $\bar{x} \in V$.*
   (b) *$(\alpha\beta) \cdot \bar{x} = \alpha \cdot (\beta \cdot \bar{x})$, for any $\alpha, \beta \in \mathbb{K}$ and $\bar{x} \in V$.*
   (c) *$(\alpha + \beta) \cdot \bar{x} = \alpha \cdot \bar{x} + \beta \cdot \bar{x}$, for any $\alpha, \beta \in \mathbb{K}$ and $\bar{x} \in V$.*
   (d) *$\alpha \cdot (\bar{x} + \bar{y}) = \alpha \cdot \bar{x} + \alpha \cdot \bar{y}$, for any $\bar{x}, \bar{y} \in V$ and $\alpha \in \mathbb{K}$.*

The elements of the vector space V are called **vectors.**

Some of the most common examples of vector spaces are:

1. $V = \mathbb{K}$ is a vector space over $\mathbb{K}$.
2. $V = \mathcal{M}_{m\times n}(\mathbb{K})$ is the vector space of all matrices $m \times n$ with entries in $\mathbb{K}$.
3. $V = \mathcal{S}_{n\times n}(\mathbb{K})$ is the vector space of all symmetric matrices $n \times n$ with entries in $\mathbb{K}$.
4. $V = \mathcal{P}_n(\mathbb{K})$ is the vector space of polynomials of degree at most n with coefficients in $\mathbb{K}$.

1.2 Properties.

1. $\alpha \cdot \bar{0} = \bar{0}$, for any $\alpha \in \mathbb{K}$.
   Proof: We have:
   $$\alpha \cdot \bar{0} = \alpha \cdot (\bar{0} + \bar{0}) = \alpha \cdot \bar{0} + \alpha \cdot \bar{0} \quad \Rightarrow \quad \alpha \cdot \bar{0} = \bar{0}.$$
2. $0 \cdot \bar{x} = \bar{0}$, for any $\bar{x} \in V$.
   Proof: We have:
   $$0 \cdot \bar{x} = (0 + 0) \cdot \bar{x} = 0 \cdot \bar{x} + 0 \cdot \bar{x} \quad \Rightarrow \quad 0 \cdot \bar{x} = \bar{0}.$$
3. $\alpha \cdot \bar{x} = \bar{0} \quad \Rightarrow \quad \alpha = 0, \quad \text{or} \quad \bar{x} = 0.$
   Proof: If $\alpha \cdot \bar{x} = \bar{0}$ and $\alpha \neq 0$ then α has an inverse and:
   $$\bar{x} = 1 \cdot \bar{x} = (\alpha^{-1}\alpha) \cdot \bar{x} = \alpha^{-1} \cdot (\alpha \cdot \bar{x}) = \alpha^{-1} \cdot \bar{0} = \bar{0}.$$
4. $(-1) \cdot \bar{x} = -\bar{x}$, for any $\bar{x} \in V$.
   Proof: It is sufficient to note that:
   $$\left.\begin{array}{r} (-1) \cdot \bar{x} + \bar{x} = (-1 + 1) \cdot \bar{x} = 0 \cdot \bar{x} = \bar{0} \\ \bar{x} + (-1) \cdot \bar{x} = (1 - 1) \cdot \bar{x} = 0 \cdot \bar{x} = \bar{0} \end{array}\right\} \quad \Rightarrow \quad -\bar{x} = (-1) \cdot \bar{x}.$$
5. $(-\alpha) \cdot \bar{x} = \alpha \cdot (-\bar{x}) = -(\alpha \cdot \bar{x})$, for any $\alpha \in \mathbb{K}$, $\bar{x} \in V$.
   Proof: From the previous properties:
   $$\begin{array}{l} (-\alpha) \cdot \bar{x} = (-1) \cdot (\alpha \cdot \bar{x}) = -(\alpha \cdot \bar{x}). \\ (-\alpha) \cdot \bar{x} = (\alpha) \cdot ((-1) \cdot \bar{x}) = \alpha \cdot (-\bar{x}). \end{array}$$
6. If $\alpha \neq 0$ and $\alpha \cdot \bar{x} = \alpha \cdot \bar{y}$ then $\bar{x} = \bar{y}$.
   Proof: If $\alpha \neq 0$ then its inverse exists and:
   $$\alpha \cdot \bar{x} = \alpha \cdot \bar{y} \quad \Rightarrow \quad (\alpha^{-1}\alpha) \cdot \bar{x} = \alpha^{-1}\alpha) \cdot \bar{y} \quad \Rightarrow \quad \bar{x} = \bar{y}.$$
7. If $\bar{x} \neq \bar{0}$ and $\alpha \cdot \bar{x} = \beta \cdot \bar{x}$ then $\alpha = \beta$.
   Proof: We have:
   $$\alpha \cdot \bar{x} = \beta \cdot \bar{x} \quad \Rightarrow \quad (\alpha - \beta) \cdot \bar{x} = \bar{0}.$$
   Since $\bar{x} = \bar{0}$, from the previous properties we deduce that:
   $$\alpha - \beta = 0 \quad \Rightarrow \quad \alpha = \beta.$$

2 Vector subspaces.

2.1 Definition and characterization.

Definition 2.1 *Given a vector space V over a field $\mathbb{K}$, a nonempty subset $S \subset V$ is said to be* **a vector subspace or linear subspace** *of V if S is a vector space over $\mathbb{K}$ under the operations of V.*

In practice, to identify vector subspaces, one usually applies one of the following characterizations:

Theorem 2.2 *Let V be a vector space over $\mathbb{K}$ and $S \subset V$ a nonempty subset. S is a linear subspace if and only if it satisfies:*

1. $\bar{x} + \bar{y} \in S$, *for any* $\bar{x}, \bar{y} \in S$.
2. $\lambda \cdot \bar{x} \in S$, *for any* $\lambda \in \mathbb{K}$, $\bar{x} \in S$.

Equivalently, if it satisfies:

(a) $\alpha \cdot \bar{x} + \beta \cdot \bar{y} \in S$, *for any* $\bar{x}, \bar{y} \in S$, $\alpha, \beta \in \mathbb{K}$.

Proof: When conditions 1 and 2 are satisfied both the internal and external operations of V restrict to S. Thus they satisfy the eight properties from Definition 1.1, and S is a vector space.

Conversely, if S is a vector space of V, the internal and external operations must restrict to S and conditions 1 and 2 hold.

Finally, let us prove the equivalence between conditions 1, 2 and condition (a).

1, 2 $\Rightarrow$ a:

By condition 2: $$\begin{array}{lcl} \bar{x} \in S \text{ e } \alpha \in \mathbb{K} & \Rightarrow & \alpha \cdot \bar{x} \in S. \\ \bar{y} \in S \text{ e } \beta \in \mathbb{K} & \Rightarrow & \beta \cdot \bar{y} \in S. \end{array}$$

Hence

By condition 1: $$\left. \begin{array}{l} \alpha \cdot \bar{x} \in S \\ \beta \cdot \bar{y} \in S \end{array} \right\} \Rightarrow \alpha \cdot \bar{x} + \beta \cdot \bar{y} \in S.$$

a $\Rightarrow$ 1 ,2:

Applying the condition (a) with $\alpha = 1, \beta = 1$ we obtain condition 1, $\bar{x} + \bar{y} \in S$.

Applying the condition (a) with $\beta = 0$ we obtain condition 2, $\alpha \cdot \bar{x} \in S$. ■

Remark 2.3 *Any vector space V contains at least two linear subspaces, called* **trivial***: the whole space V and the zero subspace $\{\bar{0}\}$.*

2.2 Intersection of linear subspaces.

Proposition 2.4 *Let S_1 and S_2 be two linear subspaces of V. Their intersection $S_1 \cap S_2$ is a linear subspace.*

Proof: First note that $S_1 \cap S_2$ is nonempty, because $\bar{0} \in S_1$ and $\bar{0} \in S_2$. Next, let us check that condition (a) holds. Let $\bar{x}, \bar{y} \in S_1 \cap S_2$ and $\alpha, \beta \in \mathbb{K}$. We have:

$$\left. \begin{array}{lclcl} \bar{x}, \bar{y} \in S_1 \cap S_2 & \Rightarrow & \bar{x}, \bar{y} \in S_1 & \Rightarrow & \alpha \cdot \bar{x} + \beta \cdot \bar{y} \in S_1 \\ \bar{x}, \bar{y} \in S_1 \cap S_2 & \Rightarrow & \bar{x}, \bar{y} \in S_2 & \Rightarrow & \alpha \cdot \bar{x} + \beta \cdot \bar{y} \in S_2 \end{array} \right\} \Rightarrow \alpha \cdot \bar{x} + \beta \cdot \bar{y} \in S_1 \cap S_2.$$

■

2.3 Sum of linear subspaces.

First, note that the union of vector subspaces need not be a vector subspace. For example, consider:

$$V = \mathbb{R}^2; \quad S_1 = \{(x, 0) \in \mathbb{R}^2, x \in \mathbb{R}\}; \quad S_2 = \{(0, y) \in \mathbb{R}^2, y \in \mathbb{R}\}.$$

Taking $(1, 0) \in S_1 \cup S_2$ and $(0, 1) \in S_1 \cup S_2$, we have $(1, 0) + (0, 1) = (1, 1) \notin S_1 \cup S_2$ which implies that the union of S_1 and S_2 is not a vector subspace.

However, given two linear subspaces we can define its sum, which will turn out to be the smallest vector subspace containing the union.

Definition 2.5 *Let S_1 and S_2 two vector subspaces of V, we define the* **sum** *of S_1 and S_2 as:*

$$S_1 + S_2 = \{\bar{x}_1 + \bar{x}_2 \text{ with } \bar{x}_1 \in S_1 \text{ and } \bar{x}_2 \in S_2\}$$

Proposition 2.6 *The sum of two subspaces is a subspace.*

Proof: Let $S_1, S_2 \in V$ be two subspaces of V. Let $\bar{x}, \bar{y} \in S_1 + S_2$ and $\alpha, \beta \in \mathbb{K}$. We have:

$$\begin{array}{lcl} \bar{x} \in S_1 + S_2 & \Rightarrow & \bar{x} = \bar{x}_1 + \bar{x}_2, \text{ with } \bar{x}_1 \in S_1, \bar{x}_2 \in S_2 \\ \bar{y} \in S_1 + S_2 & \Rightarrow & \bar{y} = \bar{y}_1 + \bar{y}_2, \text{ with } \bar{y}_1 \in S_1, \bar{y}_2 \in S_2 \end{array}$$

therefore:

$$\alpha \cdot \bar{x} + \beta \cdot \bar{y} = \alpha \cdot (\bar{x}_1 + \bar{x}_2) + \beta \cdot (\bar{y}_1 + \bar{y}_2) = \underbrace{\alpha \cdot \bar{x}_1 + \beta \cdot \bar{y}_1}_{\in S_1} + \underbrace{\alpha \cdot \bar{x}_2 + \beta \cdot \bar{y}_2}_{\in S_2} \in S_1 + S_2.$$

■

Proposition 2.7 *The sum of two linear subspaces is the smallest vector subspace containing the union.*

Proof: Let $S_1, S_2 \in V$ be two subspaces of V. First, it is clear that $S_1 \cup S_2 \subset S_1 + S_2$. Now, let S be a subspace containing $S_1 \cup S_2$; let us see that $S_1 + S_2 \subset S$.

$$\bar{x} \in S_1 + S_2 \Rightarrow \bar{x} = \bar{x}_1 + \bar{x}_2 \text{ with } \left\{ \begin{array}{l} \bar{x}_1 \in S_1 \subset S_1 \cup S_2 \subset S \\ \bar{x}_2 \in S_2 \subset S_1 \cup S_2 \subset S \end{array} \right. \Rightarrow \bar{x} = \bar{x}_1 + \bar{x}_2 \in S.$$

■

This concept can be generalized to any finite number of subspaces.

Definition 2.8 *Let $S_1, S_2, \ldots, S_k$ be subspaces of V. We define their* **sum** *as:*

$$S_1 + S_2 + \ldots + S_k = \{\bar{x}_1 + \bar{x}_2 + \ldots + \bar{x}_k \ \text{ with } \bar{x}_1 \in S_1,\ \bar{x}_2 \in S_2, \ldots,\ \bar{x}_k \in S_k\}$$

2.4 Direct sum.

Definition 2.9 *Let S_1, S_2 be two vector subspaces of U. If $S_1 \cap S_2 = \{\bar{0}\}$, then the sum subspace $S_1 + S_2$ is called* **direct sum of** S_1 **and** S_2 *and it is denoted by:*

$$S_1 \oplus S_2.$$

Proposition 2.10 *Let S_1, S_2 be two subspaces of U. The sum $S_1 + S_2$ is a direct sum if and only if any element of $S_1 + S_2$ can be uniquely written as a sum of an element of S_1 and an element of S_2.*

Proof: First, suppose the sum is direct, that is $S_1 \cap S_2 = \{\bar{0}\}$. Let us prove the uniqueness of the decomposition. Let $\bar{x} \in S_1 + S_2$. If:

$$\bar{x} = \bar{x}_1 + \bar{x}_2 = \bar{y}_1 + \bar{y}_2 \quad \text{with } \bar{x}_1, \bar{y}_1 \in S_1 \quad \text{and} \quad \bar{x}_2, \bar{y}_2 \in S_2$$

then

$$\bar{x}_1 - \bar{y}_1 = \bar{y}_2 - \bar{x}_2 \in S_1 \cap S_2 \quad \Rightarrow \quad \bar{x}_1 - \bar{y}_1 = \bar{y}_2 - \bar{x}_2 = \bar{0} \quad \Rightarrow \quad \bar{x}_1 = \bar{y}_1, \quad \bar{x}_2 = \bar{y}_2.$$

Now, assume the uniqueness of the decomposition and let us see that the sum is direct. We must prove that $S_1 \cap S_2 = \{\bar{0}\}$. Let $\bar{x} \in S_1 \cap S_2$; it can be written as:

$$\bar{x} = \bar{x} + \bar{0} \text{ with } \bar{x} \in S_1, \bar{0} \in S_2, \quad \text{and also} \quad \bar{x} = 0 + \bar{x} \text{ with } \bar{0} \in S_1, \bar{x} \in S_2.$$

By the uniqueness of the decomposition we deduce that $\bar{x} = \bar{0}$. ■

This concept can be generalized to more than two subspaces.

Definition 2.11 *Let $S_1, S_2 \ldots, S_k$ linear subspaces of U. If $(S_1 + \ldots + S_i) \cap S_{i+1} = \{0\}$ for any i, $1 \le i \le k-1$ then the sum subspace $S_1 + S_2 + \ldots + S_k$ is called* **direct sum of** $S_1, S_2, \ldots, S_k$ *and it is denoted by:*

$$S_1 \oplus S_2 \oplus \ldots \oplus S_k.$$

Proposition 2.12 *Let $S_1, S_2, \ldots, S_k$ be linear subspaces of U. The sum $S_1 + S_2 + \ldots + S_k$ is a direct sum if and only if any element of $S_1 + S_2 + \ldots + S_k$ can be written uniquely as sum of elements of $S_1, S_2, \ldots, S_k$.*

Proof: Suppose the sum is direct. Fix $\bar{x} \in S_1 + \ldots + S_k$ and suppose that we have two different decompositions of $\bar{x}$:

$$\bar{x} = \bar{x}_1 + \ldots + \bar{x}_k = \bar{y}_1 + \ldots + \bar{y}_k, \qquad \bar{x}_i, \bar{y}_i \in S_i, \quad i = 1, 2, \ldots, k.$$

Then, applying that $(S_1 + \ldots + S_{k-1}) + S_k$ is a direct sum we see that $x_k = y_k$, and:

$$\bar{x}_1 + \ldots + \bar{x}_{k-1} = \bar{y}_1 + \ldots + \bar{y}_{k-1}.$$

Now, we use that $(S_1 + \ldots + S_{k-2}) + S_{k-1}$ is a direct sum and from this $x_{k-1} = y_{k-1}$,

$$\bar{x}_1 + \ldots + \bar{x}_{k-2} = \bar{y}_1 + \ldots + \bar{y}_{k-2}.$$

Repeating the process, we deduce that $x_i = y_i$ for $i = 1, 2, \ldots, k$.

Conversely, assume the uniqueness of the decomposition. Let us see that for any $i = 2, \ldots, k$, $(S_1 + \ldots + S_i) + S_{i+1}$ is a direct sum.

Any $\bar{x} \in S_1 + \ldots + S_i$ can be written as sum of elements of $S_1, \ldots, S_i$. By the hypothesis this decomposition is unique. Applying Proposition 2.10 we deduce that the sum is direct, that is, $(S_1 + \ldots + S_i) \cap S_{i+1} = \{\bar{0}\}$. ■

2.5 Complementary subspaces.

Definition 2.13 *Let S_1, S_2 be two subspaces of V. They are called* **complementary** *if:*

$$\begin{aligned} S_1 \cap S_2 &= \{0\} \\ S_1 + S_2 &= V \end{aligned}$$

or equivalently:

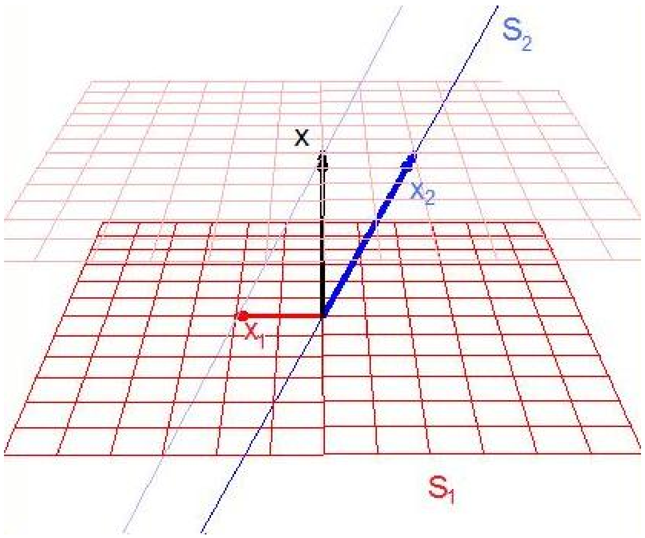
$$S_1 \oplus S_2 = V.$$

If S_1, S_2 are complementary subspaces we know that any $\bar{x} \in V$ can be written uniquely as $\bar{x} = \bar{x}_1 + \bar{x}_2$, with $\bar{x}_1 \in S_1$ and $\bar{x}_2 \in S_2$:

$\bar{x}_1$ is called **projection of** $\bar{x}$ **onto** S_1 **along** S_2.

$\bar{x}_2$ is called **projection of** $\bar{x}$ **onto** S_2 **along** S_1.

It is interesting to note that beyond the algebraic background this concept is very geometric. An example is represented in the following image. The complementary subspaces are the plane S_1 (in red) and the line S_2 (in blue); any vector $\vec{x}$ can be decomposed as the sum of a pair of vectors $\vec{x}_1$ and $\vec{x}_2$ in each of the subspaces.



Geometrically the vector $\vec{x}_1$ is constructed by taking the line parallel to S_2 through the endpoint of the vector $\vec{x}$ and intersecting it with the plane S_1. Hence it is called the projection of $\bar{x}$ onto S_1 along S_2.

Analogously, the vector $\vec{x}_2$ is constructed by intersecting the plane parallel to S_1 through the endpoint of $\vec{x}$ with the line S_1.

Based on this decomposition, we can define the following **projection functions:**

$$p_1: \begin{array}{ccc} V & \longrightarrow & V \\ \bar{x} & \longrightarrow & \bar{x}_1 \end{array} \qquad p_2: \begin{array}{ccc} V & \longrightarrow & V \\ \bar{x} & \longrightarrow & \bar{x}_2 \end{array}$$

Function p_1 is called projection function onto S_1 along S_2 and function p_2 is called projection function onto S_2 along S_1 .

These functions satisfy the following properties:

1. $p_1 + p_2 = Id$.
2. $p_1 \circ p_1 = p_1$ and $p_2 \circ p_2 = p_2$.
3. $p_1 \circ p_2 = p_2 \circ p_1 = 0$.