4. Systems of linear equations.

1 Definitions and matrix representation.

1.1 Basic definitions.

Definition 1.1 *A system of m linear equations of n unknowns $x_1, \ldots, x_n$ with coefficients in a field $\mathbb{K}$ is a set of equations of the form:*

$$\left.\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m \end{array}\right\} m \text{ equations}$$

where $a_{ij}, b_i \in \mathbb{K}$.

If all constant terms $b_1, \ldots, b_m$ are zero, the system is said to be **homogeneous.**

A linear system is said to be **consistent** if it has a solution, that is, if there are scalars $x_1, \ldots, x_n \in \mathbb{K}$ simultaneously satisfying its m equations. In other case it is called **inconsistent**.

When the system is consistent, it can be **determined** when the solution is unique or **undetermined** when there are more than one solution.

1.2 Matrix representation.

Given a linear system, we can write it as a matrix equation :

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{B}$$

We call:

- $A \in \mathcal{M}_{m\times n}(\mathbb{K})$ **coefficient matrix** of the system or simply **matrix of the system**.

- $X \in \mathcal{M}_{n\times 1}(\mathbb{K})$ **matrix (or column vector) of unknowns**.

- $B \in \mathcal{M}_{m\times 1}(\mathbb{K})$ **matrix (or column vector) of constants**.

Moreover we denote by $\bar{A} \in \mathcal{M}_{m\times(n+1)}(\mathbb{K})$ the matrix obtained by appending the column vector of constants to the coefficient matrix A; it is called the **augmented matrix of the system**:

$$\bar{A} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_n \end{array}\right).$$

2 Existence of solution: Rouché-Fröbenius Theorem.

Theorem 2.1 (Rouché-Fröbenius Theorem) *A linear system of m linear equations and n unknowns is consistent if and only if the rank of the matrix system is equal to the rank of the augmented matrix system.*

Proof: Consider the matrix expression of the linear system $AX = B$. The existence of solution is equivalent to the existence of scalars $x_1, \ldots, x_n$ such that:

$$\sum_{k=1}^{n} a_{ik}x_k = b_i \qquad \text{for } i = 1, \ldots, m.$$

If we denote the columns of the matrix A by A_i, this relation can be written as

$$\sum_{k=1}^{n} x_k A_k = B.$$

This means that the vector B is a linear combination of the columns of matrix A.

Therefore, the linear system is consistent if and only if the column B is a linear combination of the columns of A. Since the rank of a matrix is the number of linearly independent columns it has (see Theorem 4.2, Chapter 2, Tema III), the result follows. ■

3 Equivalent systems and resolution methods.

3.1 Equivalent systems.

Definition 3.1 *Two linear systems with n unknowns are said to be* **equivalent** *if they have exactly the same solutions.*

Theorem 3.2 *Let $AX = B$ and $A'X = B'$ two linear systems of m equations and n unknowns. If $\bar{A}$ and $\bar{A}'$ are row equivalent, then the systems are equivalent.*

Proof: If $\bar{A}$ and $\bar{A}'$ are row equivalent, there exists a nonsingular matrix $P \in \mathcal{M}_{m\times m}$ such that

$$\bar{A}' = P\bar{A}$$

adn equivalently

$$(A'|B') = P(A|B) \iff (A'|B') = (PA|PB) \iff A' = PA, \qquad B' = PB.$$

Thus, given a solution X of the first system, that is, a column satisfying $AX = B$ we have:

$$A'X = PAX = PB = B'$$

and therefore X is also a solution of the second system. Conversely, given a solution X of the second system ($A'X = B'$) we have:

$$AX = P^{-1}A'X = P^{-1}B' = B$$

We see that X is a solution of the first system. We conclude that both systems have the same solutions. ■

Remark 3.3 *Given a linear system of equations $AX = B$, by the previous theorem we know that by performing elementary row operations on the equations on the augmented matrix $\bar{A}$, we obtain a new linear system with the same solutions.*

It is immediate that doing elementary row operations to $\bar{A}$ is equivalent to doing operations to the equations of the system. That is, carrying out any of the following transformations on the equations of a linear system provides an equivalent linear system (with an identical set of solutions):

- *Multiply an equation by a non-zero number.*
- *Swap the positions of two equations.*
- *Add to an equation a multiple of a different one.*

3.2 Gaussian elimination method.

3.2.1 Description of the method.

The idea of the Gaussian elimination method is to apply row elementary operations with the aim to obtain an equivalent system which is simpler than the original one. Specifically, we set to find an echelon form of the matrix system with 1's as the leading elements of all nonzero rows.

The steps to solve a linear system by Gaussian elimination method are:

1. We consider the augmented matrix $\overline{A}$ of the system.
2. We reduce this matrix by applying elementary row operations until it becomes a row echelon matrix:
   (a) If the matrix is composed entirely of zeros, then we are done.
   (b) Otherwise, starting from the left, we look for the first column with some nonzero element a. We move the row in which it appears to the first position.
   (c) We divide the first row by a so that the leading element of the first row is 1.
   (d) Using the first row and applying elementary row operations, we get zeros below the leading element of the first row.
   (e) We repeat the process on the submatrix formed by the remaining rows. We are done when we have simplified all nonzero rows.
3. We have a new linear system equivalent to the initial one but now the augmented matrix $\overline{A}'$ is a row echelon matrix.
4. If there is any row all whose elements are zeros except for the last one (which corresponds to the constant term) the system is inconsistent.
5. Otherwise, the system is consistent. We move to the right-hand side of each equation all the terms from columns with no leading entries; we will then have a system with $rank(A)$ unknowns and $n - rank(A)$ parameters. This system can be easily solved by succesive substitution, starting with the last equation and ending with the first.

3.2.2 Example of Gaussian elimination.

Suppose we want to solve the linear system

$$\begin{array}{rrrrcl} & 3y & -z & & = & 6 \\ x & +y & +z & +t & = & 5 \\ 2x & -y & +3z & +2t & = & 4 \end{array}$$

The augmented matrix of the system is

$$\bar{A} = \left(\begin{array}{rrrr|r} 0 & 3 & -1 & 0 & 6 \\ 1 & 1 & 1 & 1 & 5 \\ 2 & -1 & 3 & 2 & 4 \end{array}\right)$$

We apply row elementary operations until we obtain a row echelon matrix:

$$\bar{A} \xrightarrow{H_{12}} \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 5 \\ 0 & 3 & -1 & 0 & 6 \\ 2 & -1 & 3 & 2 & 4 \end{array}\right) \xrightarrow{H_{31}(-2)} \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 5 \\ 0 & 3 & -1 & 0 & 6 \\ 0 & -3 & 1 & 0 & -6 \end{array}\right) \xrightarrow{H_{32}(1)}$$

$$\xrightarrow{H_{32}(1)} \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 5 \\ 0 & 3 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{H_{2}(1/3)} \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -1/3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We have obtained the equivalent linear system

$$\begin{array}{rrrrcl} x & +y & +z & +t & = & 5 \\ & y & -\frac{1}{3}z & & = & 2 \end{array}$$

No row has only the constant term different from zero, therefore the system is consistent. We move to the right-hand side of each equation all the terms from columns with no leading elements

$$\begin{array}{rrcl} x & +y & = & 5 - z - t \\ & y & = & 2 + \frac{1}{3}z \end{array}$$

We will obtain the solution as a function of two parameters (the linear system is consistent but undetermined).

From the second equation:

$$y = 2 + \frac{1}{3}z.$$

and substituting in the first:

$$x = 5 - z - t - y = 5 - z - t - 2 - \frac{1}{3}z = 3 - \frac{4}{3}z - t.$$

3.3 Cramer's rule.

Theorem 3.4 (Cramer's rule) *Given a consistent and determined linear system $AX = B$ of n equations and n unknowns, its solution can be obtained as*

$$x_i = \frac{det(M_i)}{det(A)} \qquad \textit{for} \quad i = 1, \ldots, n,$$

where M_i is the matrix obtained by replacing the i-th column of A by the constant vector B.

Proof: Since the system is consistent and determined, $rank(A) = n$ and thus A is invertible. Hence

$$AX = B \;\Rightarrow\; A^{-1}AX = A^{-1}B \;\Rightarrow\; X = A^{-1}B.$$

Now, let us recall that A^{-1} can be obtanied as:

$$A^{-1} = \frac{adj(A)}{det(A)}.$$

Then:

$$x_i = \sum_{k=1}^{n} (A^{-1})_{ik} B_k = \frac{1}{det(A)} \sum_{k=1}^{n} A_{ki} B_i$$

But this is the cofactor expansion of the determinant along the i-th column of the matrix M_i. ■

Corollary 3.5 *Let $AX = B$ be a consistent linear system with m equations and n unknowns; then the set of solutions depends on $n - rank(A)$ parameters.*

Proof: By the Rouché-Fröbenius theorem, since it is a consistent system, $rank(\bar{A}) = rank(A)$. This means that there are $rank(A)$ independent equations. Each of the remaining ones can be expressed as a linear combination of them. Therefore by doing elementary operations we can eliminate them and obtain an equivalent system $A'X = B'$ with exactly $rank(A) = rank(A')$ equations.

Now, by the definition of rank, there is a minor of A' with nonzero determinant; this minor is obtained by choosing $rank(A')$ columns of the matrix. Each column corresponds to the coefficients multiplying one of the unknowns x_i. Moving the remaining unknowns to the right-hand member and regarding them as parameters, we have a consistent determined linear system of $rank(A')$ equations. This system can be solved by the Cramer's rule. ■

4 Discussion of a system of linear equations.

We summarize the results about existence and uniqueness of solution of a linear system in the following table:

Given a linear system of m equations and n unknowns

$$\left.\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m \end{array}\right\} m \text{ equations}$$

the following holds:

1. If $rank(A) = rank(\bar{A})$ the system is consistent. Moreover:
   (a) if $rank(A) = rank(\bar{A}) = n$, the system is determined.
   (b) if $rank(A) = rank(\bar{A}) < n$, the system is undetermined and the solution depends on $n - rank(A)$ parameters.
2. If $rank(A) < rank(\bar{A})$ then the system is inconsistent.