# 3. Equivalence and congruence of matrices.

## 1 Elementary transformations.

### 1.1 Elementary row operations.

The three types of **elementary row operations** are:

1. $H_{ij}$ : Interchanging $i$-th row with $j$-th row.
2. $H_i(\lambda)$ : The $i$-th row is multiplied by the scalar $\lambda \neq 0$.
3. $H_{ij}(\lambda)$ : The $j$-th row is multiplied by $\lambda$ and added to $i$-th row.

### 1.2 Elementary row matrices.

An **elementary row matrix** is the result of applying an elementary row operation to the identity matrix of a given dimension. Thus there are three types of elementary row matrices:

1. $H_{ij}$ : Matrix obtained by exchanging rows $i, j$ of the identity matrix.
2. $H_i(\lambda)$ : Matrix obtained by multiplying row $i$ of the identity matrix by $\lambda \neq 0$.
3. $H_{ij}(\lambda)$ : Matrix obtained from the identity matrix when the $j$-th row is multiplied by $\lambda$ and added to the $i$-th row.

**Theorem 1.1** *Performing an elementary* **row** *operation on a matrix $A \in \mathcal{M}_{m\times n}$ is equivalent to multiplying $A$* **on the left** *by the corresponding elementary row matrix of dimension $m$.*

**Proof:**

1. Let us compute $B = H_{\alpha\beta} \cdot A$:

$$b_{ij} = \sum_{k=1}^{m} (H_{\alpha\beta})_{ik} a_{kj} = \begin{cases} a_{ij} & \text{if } i \neq \alpha,\ i \neq \beta. \\ a_{\beta j} & \text{if } i = \alpha. \\ a_{\alpha j} & \text{if } i = \beta. \end{cases}$$

   We see that the matrix $B$ is obtained from $A$ by exchanging rows $\alpha, \beta$.

2. Let us compute $B = H_\alpha(\lambda) \cdot A$:

$$b_{ij} = \sum_{k=1}^{m} H_\alpha(\lambda)_{ik} a_{kj} = \begin{cases} a_{ij} & \text{if } i \neq \alpha. \\ \lambda a_{\alpha j} & \text{if } i = \alpha. \end{cases}$$

   We see that the matrix $B$ is obtained from $A$ by multiplying row $\alpha$ by $\lambda$.

3. Let us compute $B = H_{\alpha\beta}(\lambda) \cdot A$:

$$b_{ij} = \sum_{k=1}^{m} H_{\alpha\beta}(\lambda)_{ik} a_{kj} = \begin{cases} a_{ij} & \text{if } i \neq \alpha. \\ a_{ij} + \lambda a_{\beta j} & \text{if } i = \alpha. \end{cases}$$

   We see that the matrix $B$ is obtained from $A$ by adding to row $\alpha$ row $\beta$ multiplied by $\lambda$.

**Proposition 1.2** *All elementary row matrices are invertible. In particular:*

$$(H_{ij})^{-1} = H_{ij}; \qquad (H_i(\lambda))^{-1} = H_i(\frac{1}{\lambda}); \qquad (H_{ij}(\lambda))^{-1} = H_{ij}(-\lambda);$$

**Proof:** It is sufficient to note the following facts:

- The inverse elementary operation of exchanging rows $i, j$ is swapping back these rows.

- The inverse elementary operation of multiplying row $i$ by $\lambda$ is dividing it by $\lambda$.

- The inverse elementary operation of adding to row $i$ row $j$ multiplied by $\lambda$, is adding to row $i$ row$j$ multiplied by $-\lambda$.

### 1.3 Elementary column operations.

The three types of **elementary column operations** are:

1. $V_{ij}$ : Interchanging $i$-th column with $j$-th column.
2. $V_i(\lambda)$ : The $i$-th column is multiplied by the scalar $\lambda \neq 0$.
3. $V_{ij}(\lambda)$ : The $j$-th column is multiplied by $\lambda$ and added to the $i$-th column.

### 1.4 Elementary column matrix.

An **elementary column matrix** is the result of applying an elementary column operation on the identity matrix of a given dimension. Thus there are three types of elementary column matrices:

1. $V_{ij}$ : Matrix obtained by exchanging columns $i, j$ of the identity matrix.
2. $V_i(\lambda)$ : Matrix obtained by multiplying column $i$ of the identity matrix by $\lambda \neq 0$.
3. $V_{ij}(\lambda)$ : Matrix obtained from the identity matrix when the $j$-th column is multiplied by $\lambda$ and added to column $i$-th.

It is clear that

$$V_{ij} = (H_{ij})^t; \qquad V_i(\lambda) = (H_i(\lambda))^t; \qquad V_{ij}(\lambda) = (H_{ij}(\lambda))^t;$$

As a consequence of this, elementary column matrices satisfy similar properties to those of the elementary row matrices.

**Theorem 1.3** *Performing an elementary* **column** *operation on a matrix $A \in \mathcal{M}_{m\times n}$ is equivalent to multiplying $A$* **on the right** *by the corresponding elementary column matrix of dimension $m$.*

**Proposition 1.4** *All elementary column matrices are invertible. In particular:*

$$(V_{ij})^{-1} = V_{ij}; \qquad (V_i(\lambda))^{-1} = V_i(\frac{1}{\lambda}); \qquad (V_{ij}(\lambda))^{-1} = V_{ij}(-\lambda);$$

# 2 Row equivalence of matrices.

## 2.1 Definition and properties.

**Definition 2.1** *Two matrices $A, B \in \mathcal{M}_{m\times n}$ are said to be* **row equivalent** *or* **left equivalent** *if one of them can be obtained from the other by applying a sequence of elementary row operations:*

$$A \text{ is row equivalent to } B \iff B = H_p \cdot H_{p-1} \cdot \ldots H_1 \cdot A.$$

Let us see some properties:

1. *Row equivalence satisfies reflexive, symmetric and transitive properties.*
   - Reflexive (any matrix is row equivalent to itself ).
   - Symmetric (if $A$ is row equivalent to $B$, then $B$ is row equivalent to $A$):

   $$B = H_p \cdot H_{p-1} \cdot \ldots \cdot H_1 \cdot A \quad \Rightarrow \quad A = H_1^{-1} \cdot \ldots \cdot H_{p-1}^{-1} \cdot H_p^{-1} \cdot B.$$

   Since the inverse of a row elementary operation is a row elementary operation, we deduce that $B$ is row equivalent to $A$.
   - Transitive (if $A$ is row equivalent to $B$, and $B$ is row equivalent to $C$ then $A$ is row equivalent to $C$): If $A$ and $B$ are row equivalent and $B$ and $C$ are row equivalent, we have:

   $$\left.\begin{array}{l} B = H_p \cdot H_{p-1} \cdot \ldots \cdot H_1 \cdot A \\ C = H'_q \cdot H'_{q-1} \cdot \ldots \cdot H'_1 \cdot B \end{array}\right\} \Rightarrow C = H'_q \cdot H'_{q-1} \cdot \ldots \cdot H'_1 \cdot H_p \cdot H_{p-1} \cdot \ldots \cdot H_1 \cdot A$$

   and therefore $A$ and $C$ are row equivalent.
2. *Two row equivalent matrices have the same dimension.*
3. *Two row equivalent matrices have the same rank.*
   **Proof:** We have seen that the rank of a matrix does not change if we perform elementary transformations on a matrix.
   **Remark:** Two matrices with the same rank do not have to be row equivalent. For example:

   $$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

## 2.2 Reduced row echelon form.

**Definition 2.2** *Given a matrix $A \in \mathcal{M}_{m\times n}$ the first nonzero element of each row is called* **leading element or pivot.**

**Definition 2.3** *A matrix $A \in \mathcal{M}_{m\times n}$ is in* **row echelon form** *if the leading element of each row is always strictly to the right of the leading element of the row above it.*

**Definition 2.4** *A* **reduced row echelon form** *is a matrix $R \in \mathcal{M}_{m\times n}$ satisfying the following properties:*

1. *It is a row echelon form.*
2. *The leading element of each nonzero row is 1.*
3. *Each column containing a leading element 1 has zeros as all its other entries.*

Some examples of reduced row echelon form are (the leading elements of each row are in red):

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The reduced row echelon form of a matrix $A \in \mathcal{M}_{m\times n}$ is a reduced row echelon form row equivalent to $A$. Roughly speaking, it is the simplest matrix equivalent by rows to the given matrix. It can be proved that the reduced row echelon form of a matrix $A$ is unique. From this, we deduce the following result:

**Theorem 2.5** *Two matrices $A, B \in \mathcal{M}_{m\times n}$ are row equivalent if and only if they have the same reduced row echelon form.*

## 2.3 Calculation of the reduced row echelon form of a given matrix.

Given a matrix $A \in \mathcal{M}_{m\times n}$, it consists in applying elementary row operations in order to obtain its reduced row echelon form. To do this, the columns of the matrix $A$ will be successively simplified. The procedure is as follows:

1. If there is a nonzero element in the first column, move it to position $1,1$ by changing rows: $H_{1j}$. If all elements are null, go to the next column.
2. Now the element $a'_{1,1}$ at position $1,1$ is turned into 1 by dividing the whole row by it. The corresponding elementary operation is $H_1(\frac{1}{a'_{11}})$.
3. We get zeros on the first column. To do this, the operations $H_{j1}(-a'_{j1})$ are successively performed.

4. The analogous process is repeated with the next column, taking into account that whenever we get a column with one 1 and zeros elsewhere, it will not be modified further.

For example, if $A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & -2 & 1 \\ 2 & -1 & 4 \end{pmatrix}$ let us compute its reduced row echelon form:

$$\begin{pmatrix} 0 & 1 & 3 \\ 2 & -2 & 1 \\ 2 & -1 & 4 \end{pmatrix} \overset{H_{12}}{\longrightarrow} \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 3 \\ 2 & -1 & 4 \end{pmatrix} \overset{H_1(1/2)}{\longrightarrow} \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & 3 \\ 2 & -1 & 4 \end{pmatrix} \longrightarrow$$

$$\overset{H_{31}(-2)}{\longrightarrow} \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{pmatrix} \overset{H_{12}(-1)}{\longrightarrow} \begin{pmatrix} 1 & 0 & 7/2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{pmatrix} \overset{H_{32}(-1)}{\longrightarrow} \begin{pmatrix} 1 & 0 & 7/2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

### 2.4 Row equivalence of a nonsingular square matrix.

**Theorem 2.6** *If $A$ is an $n$-dimensional nonsingular square matrix then it is row equivalent to the identity matrix.*

**Proof:** Just apply the process described in the previous section to the matrix $A$. Since $A$ is regular, all matrices which are row equivalent to it are also regular. Therefore, on each of the steps we take during this reduction, neither a row of zeros nor a column of zeros can ever appear. As a consequence of this, at the end of the reduction process, we will reach the identity matrix. ■

**Corollary 2.7** *If $A$ is an $n$-dimensional nonsingular square matrix then it can be decomposed into the product of elementary row matrices.*

**Proof:** By the previous theorem, $Id$ and $A$ are row equivalent, so:

$$A = H_p \cdot H_{p-1} \cdot \ldots \cdot H_1 \cdot Id = H_p \cdot H_{p-1} \cdot \ldots \cdot H_1.$$

■

**Corollary 2.8** *Two matrices $A, B \in \mathcal{M}_{m\times n}$ are row equivalent if and only if there exists a regular $P \in \mathcal{M}_{m\times m}$ such that:*

$$B = PA.$$

**Proof:** From the definition, it is clear that if $A$ and $B$ are row equivalent the matrix $P$ exists: it is the product of all elementary row matrices corresponding to the elementary row transformations which turn $A$ into $B$. Conversely, if the matrix $P$ exists, applying the previous corollary we know that it can be decomposed into elementary row transformations, so:

$$B = PA \quad \Rightarrow \quad B = H_p \cdot H_{p-1} \cdot \ldots \cdot H_1 \cdot A \quad \Rightarrow \quad A, B \text{ row equivalent.}$$

■

**Remark 2.9** *If we perform row operations on a matrix $A$ until we reach another matrix $B$, we can find out which is the matrix $P$ satisfying $B = PA$, in two ways:*

*1) Doing to the identity the same row transformations that we did to $B$.*

*2) Going through the entire process at the same time. To do this, we place the identity matrix next to $A$ and perform row operations on the extended matrix:*

$$(\,A\,|\,I\,) \longrightarrow \textit{Row operations} \longrightarrow (\,B\,|\,P\,)$$

### 2.5 Inverse of a matrix using elementary row operations (Gauss-Jordan method).

Applying the results about row equivalence, we can calculate the inverse of a nonsingular square matrix as follows:

We place the identity matrix to the right of matrix $A$. We reduce the matrix $A$ to the identity matrix with row operations, following the method previously described. Finally, on the right we will have the inverse of matrix $A$:

$$(\,A\,|\,I\,) \longrightarrow \text{Row operations} \longrightarrow (\,I\,|\,A^{-1}\,).$$

## 3 Column equivalence of matrices.

**Definition 3.1** *Two matrices $A, B \in \mathcal{M}_{m\times n}$ are said to be* **column equivalent** *or* **right equivalent** *if one of them can be obtained from the other by applying a sequence of elementary column operations:*

$$A, B \textit{ column equivalent} \iff B = A \cdot V_1 \cdot V_2 \cdot \ldots \cdot V_p.$$

Since transposition turns column operations into row operations, all the properties shown for row equivalence are also satisfied by column equivalence.

1. *Column equivalence satisfies reflexive, symmetric and transitive properties.*
2. *Two column equivalent matrices have the same dimension.*
3. *Two column equivalent matrices have the same rank.* The converse does not have to be true.

### 3.1 Reduced column echelon form.

**Definition 3.2** *Given a matrix $A \in \mathcal{M}_{m\times n}$ the first nonzero element of each column is called the* **leading element or pivot** *of the column.*

**Definition 3.3** *A matrix $A \in \mathcal{M}_{m\times n}$ is in* **column echelon form** *if the leading element of each column is always strictly lower than the leading coefficient of the previous column.*

**Definition 3.4** *A* **reduced column echelon form** *is a matrix $R \in \mathcal{M}_{m\times n}$ satisfying the following properties:*

1. *It is a column echelon form.*
2. *The leading element of each nonzero column is 1.*
3. *Each row containing a leading element 1 has zeros as all its other entries.*

Some examples of reduced column echelon form are (the leading elements of each column are in red):

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The reduced column echelon form of a matrix $A \in \mathcal{M}_{m\times n}$ is a reduced column echelon form column equivalent to $A$. Roughly speaking, it is the simplest matrix equivalent by columns to the given matrix. It can be prove that the column echelon form of a matrix $A$ is unique. From this, we deduce the following result:

**Theorem 3.5** *Two matrices $A, B \in \mathcal{M}_{m\times n}$ are column equivalent if and only if they have the same reduced column echelon form.*

## 3.2 Calculation of the reduced column echelon form of a given matrix.

Given a matrix $A \in \mathcal{M}_{m\times n}$, it consists in applying elementary column operations in order to obtain its reduced column echelon form. To do this, the rows of the matrix $A$ will be successively simplified. The procedure is as follows:

1. If there is a nonzero element in the first row, move it to position $1,1$ by changing columns: $V_{1j}$. If all elements are null, go to the next row.
2. Now the element $a'_{1,1}$ at position $1,1$ is turned into 1 by dividing the whole column by it. The corresponding elementary operation is $V_1(\frac{1}{a'_{11}})$.
3. We get zeros on the first row. To do this, the operations $V_{j1}(-a'_{j1})$ are successively performed.
4. The analogous process is repeated with the next row, taking into account that whenever we get a row with one 1 and zeros elsewhere, it will not be modified further.

For example, if $A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & -2 & 1 \\ 2 & -1 & 4 \end{pmatrix}$ let us see what its reduced column echelon form is:

$$\begin{pmatrix} 0 & 1 & 3 \\ 2 & -2 & 1 \\ 2 & -1 & 4 \end{pmatrix} \xrightarrow{V_{12}} \begin{pmatrix} 1 & 0 & 3 \\ -2 & 2 & 1 \\ -1 & 2 & 4 \end{pmatrix} \xrightarrow{V_{31}(-3)} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 7 \\ -1 & 2 & 7 \end{pmatrix} \longrightarrow$$

$$\xrightarrow{V_2(1/2)} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 7 \\ -1 & 1 & 7 \end{pmatrix} \xrightarrow{V_{12}(2)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 7 \\ 1 & 1 & 7 \end{pmatrix} \xrightarrow{H_{32}(-7)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

## 3.3 Column equivalence of a nonsingular square matrix.

Analogously to the case of row equivalence, the following results can be proved:

**Theorem 3.6** *If $A$ is an $n$-dimensional nonsingular square matrix then it is column equivalent to the identity matrix.*

**Corollary 3.7** *If $A$ is an $n$-dimensional nonsingular square matrix then it can be decomposed into the product of elementary column matrices.*

**Corollary 3.8** *Two matrices $A, B \in \mathcal{M}_{m\times n}$ are column equivalent if and only if there exists a regular $Q \in \mathcal{M}_{m\times m}$ such that*

$$B = AQ.$$

If $A$ is a regular square matrix we can calculate its inverse by using column operations:

$$\left(\frac{A}{I}\right) \longrightarrow \text{Column operations} \longrightarrow \left(\frac{I}{A^{-1}}\right).$$

# 4 Matrix equivalence.

## 4.1 Definition and properties.

**Definition 4.1** *Two matrices $A, B$ are said to be* **equivalent** *if one of them can be obtained from the other by applying a sequence of elementary column and/or row operations:*

$$B = H_p \cdot \ldots \cdot H_1 \cdot \ldots \cdot A \cdot V_1 \cdot \ldots \cdot V_q.$$

**Theorem 4.2** *Two matrices $A, B$ are* **equivalent** *if and only if there are nonsingular matrices $P, Q$ such that:*

$$B = PAQ.$$

**Proof:** It is a direct consequence of the analogous theorems for row and column equivalence. ■

Matrix equivalence satisfies the following properties:

1. *Matrix equivalence satisfies reflexive, symmetric and transitive properties.*
2. *Two equivalent matrices have the same dimension.*
3. *Two equivalent matrices have the same rank.*

   **Remark:** As we will see later, in this case the reciprocal is true when the matrices have the same dimension. That is, **two matrices are equivalent if and only if they have the same rank and dimension**.

## 4.2 Reduced forms in equivalence.

**Theorem 4.3** *Let $A \in \mathcal{M}_{m\times n}$ be any matrix. A is equivalent to one of the following matrices:*

$$I_r; \quad or \quad \begin{pmatrix} I_r & \Omega \\ \Omega & \Omega \end{pmatrix}; \quad or \quad \begin{pmatrix} I_r & \Omega \end{pmatrix}; \quad or \quad \begin{pmatrix} I_r \\ \Omega \end{pmatrix}.$$

*where $r = rank(A)$. These matrices are called* **canonical forms** *of matrix equivalence.*

**Corollary 4.4** *Two matrices A and B are equivalent if and only if they have the same* **rank** *and dimension.*

# 5 Congruence of square matrices.

## 5.1 Definition and properties.

**Definition 5.1** *Two square matrices A and B are* **congruent** *if A can be turned into B by a sequence of pairs of elementary operations, each pair consisting of an elementary row operation followed by the same elementary column operation*

$$B = H_p \cdot \ldots \cdot H_1 \cdot A \cdot V_1 \cdot \ldots \cdot V_p \qquad with\ V_i = H_i^t.$$

**Theorem 5.2** *Two square matrices A and B are* **congruent** *if and only if there exists a regular square matrix Q satisfying:*

$$B = Q^t \cdot A \cdot Q.$$

**Proof:** It follows from the fact that any regular matrix $Q$ can be decomposed in the product of elementary column matrices. The transpose matrix $Q^t$ then decomposes into the elementary row matrices corresponding to the same transformations. ■

Matrix congruence satisfies the following properties:

1. *Matrix congruence satisfies reflexive, symmetric and transitive properties.*
2. *Any two congruent matrices are equivalent.*
3. *Any two congruent matrices have the same dimension.*
4. *The determinants of any two congruent matrices have the same sign.*

   **Proof:** If $A$ and $B$ are congruent, there exits a regular matrix $Q$ with:

   $$B = Q^t AQ \quad \Rightarrow \quad det(B) = det(Q^t)det(A)det(Q) = det(A)det(Q)^2.$$

   In $\mathbb{R}$, $det(Q)^2 > 0$ and therefore $det(A)$ and $det(B)$ have the same sign.
5. *Any two congruent matrices have the same rank.*

   **Remark:** The converse is not true. For example the following matrices have both rank 2 but they are not congruent:

   $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

## 5.2 Congruence of symmetric matrices.

**Theorem 5.3** *Any matrix which is congruent with a symmetric matrix is itself symmetric.*

**Proof:** Suppose $A$ is symmetric and $B$ is congruent with $A$. Then there exists a nonsingular matrix $Q$ with

$$B = Q^t AQ.$$

Let us see that $B$ is symmetric, that is, it satisfies $B^t = B$:

$$B^t = (Q^t AQ)^t = Q^t A^t Q = Q^t AQ = B.$$

■

**Theorem 5.4** *A necessary and sufficient condition for a square matrix A to be diagonalizable by congruence is to be symmetric.*

**Proof:**

"$\Longrightarrow$" If $A$ is diagonalizable by congruence, then there exists a diagonal matrix $D$ congruent with $A$. Since $D$ is symmetric, so is $A$.

"$\Longleftarrow$" Suppose that $A$ is symmetric. The steps to diagonalize it by congruence are:

1. If some element of the diagonal is not zero (for example $a_{kk}$), we move it to position $1, 1$ by the operations $H_{1k}$ and $V_{1k}$.

   If all the diagonal elements of $A$ are zero, then we choose any other nonzero element at any position $i, j$. We apply the operations $H_{ji}(1)$ and $V_{ji}(1)$ so that at position $j, j$ we will get a nonzero element. Now we can apply the previous step.

If all the elements of $A$ are zero, we are done.

2. We use the nonzero element of $A$ to make zeros in the first column and in the first row . We apply the operations $H_{i1}(\frac{-a_{i1}}{a_{11}})$ and $V_{i1}(\frac{-a_{i1}}{a_{11}})$, for $i > 1$. Since **the matrix is symmetric** we get zeros in the first row and column except at position $1, 1$.
3. Now we repeat the same process. The rows and columns that have already been reduced are not modified again, so they are not used in the next steps.
4. At the end of the process we will have obtained a diagonal matrix $D$ congruent with $A$. ■

**Remark 5.5** *In the diagonalization process previously described, we can obtain the matrix $Q^t$ that allows us to go from $A$ to the diagonal $D$:*

$$D = Q^t AQ.$$

*The idea is the same as in reduction by equivalence:*

$$(\, A \,|\, I \,) \longrightarrow \text{ reduction by congruence } \longrightarrow (\, D \,|\, Q^t \,)\,.$$

### 5.3 Canonical forms of symmetric matrices.

**Theorem 5.6** *Let $A$ be a symmetric square matrix over the field of* **complex numbers**. *Then $A$ is congruent with a matrix of the form:*

$$\left(\begin{array}{c|c} I_r & \Omega \\ \hline \Omega & \Omega \end{array}\right).$$

*where $r = rank(A)$. This matrix is called* **canonical form by congruence in $\mathbb{C}$.**

**Proof:** We have seen that any square symmetric matrix is congruent with a diagonal matrix $D$. But we can still do the following reduction: For each non-zero element $d_{ii}$ of $D$ we perform the row and column operations $H_i(\frac{1}{\sqrt{d_{ii}}})$ and $V_i(\frac{1}{\sqrt{d_{ii}}})$. It can always be done in the **field of complex numbers** beacuse in $\mathbb{C}$ square roots always exist. In this way we obtain a congruent matrix with $A$ as described in the statement of the theorem. ■

**Theorem 5.7** *Let $A$ be a symmetric square matrix over the field of* **real numbers**. *Then $A$ is congruent with a matrix of the form:*

$$\left(\begin{array}{c|c|c} I_p & \Omega & \Omega \\ \hline \Omega & -I_q & \Omega \\ \hline \Omega & \Omega & \Omega \end{array}\right),$$

*where $p + q = rank(A)$, $p, q \geq 0$. This matrix is called* **canonical form by congruence in $\mathbb{R}$.**

**Proof:** We proceed as in the previous theorem. Now, the difference is that there are not real square roots of negative numbers. Therefore, for each non-zero element $d_{ii}$ of $D$ we make the row and column operations $H_i(\frac{1}{\sqrt{|d_{ii}|}})$ and $V_i(\frac{1}{\sqrt{|d_{ii}|}})$. In this way we obtain a congruent matrix with $A$ as described in the statement of the theorem ■

# 6 Similarity of square matrices.

**Definition 6.1** *Two square matrices $A$ and $B$ are said to be* **similar** *if and only if there exists a nonsingular matrix $P$ such that:*

$$B = P^{-1} \cdot A \cdot P.$$

Matrix similarity satisfies the following properties:

1. *Matrix similarity satisfies reflexive, symmetric and transitive properties.*
   - Reflexive. It is clear that any matrix is similar to itself (taking $P = Id$).
   - Symmetric. If $A$ is similar to $B$, there is a nonsingular matrix $P$ with:

   $$B = P^{-1}AP \quad \Rightarrow \quad A = (P^{-1})^{-1}BP.$$

   We see that $B$ is also similar to $A$.
   - Transitive: If $A$ and $B$ are similar, and $B$ and $C$ are also similar, then there exist regular matrices $P, P'$ with

   $$\left.\begin{array}{r} B = P^{-1}AP \\ C = P'^{-1}BP' \end{array}\right\} \quad \Rightarrow \quad C = P'^{-1}P^{-1}APP' = (PP')^{-1}BPP'$$

   where $PP'$ is nonsingular. Hence $A$ and $C$ are similar.
2. *Any two similar matrices are equivalent.*
3. *Any two similar matrices have the same dimension.*
4. *Any two similar matrices have the same determinant.*
   **Proof:** If $B = P^{-1}AP$:

   $$|B| = |P^{-1}AP| = |P|^{-1}|A||P| = |A|.$$

5. *Any two similar matrices have the same rank.*
   **Remark:** The converse is not true. For example, the following matrices have rank 2 but they are not similar because they have different determinants:

   $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

# 7 Appendix: How to reduce a matrix to its echelon form.

## 7.1 General formula:

Suppose we have a matrix:

$$\begin{matrix} \\ i \rightarrow \\ \\ j \rightarrow \\ \\ \end{matrix} \begin{pmatrix} * & * & * & \dots & * \\ p & * & * & \dots & * \\ * & * & * & \dots & * \\ q & * & * & \dots & * \\ * & * & * & \dots & * \end{pmatrix}$$

We can turn the element $q$ at row $j$ into a zero by using the element $p$ at row $i$. We apply the following operation:

$$\boxed{H_{ji}\left(\frac{-q}{p}\right)}$$

The element $p$ that we use to make zeros below is called **pivot**.

## 7.2 Example.

We are going to reduce the following matrix to the echelon form:

$$A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Fist, we want to obtain a zero at row 2, column 1 by using the element at row 1, column 1:

$$\begin{pmatrix} \boxed{2} & 4 & 1 \\ \boxed{3} & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

We perform the operation $H_{21}\left(\frac{-3}{2}\right)$ and obtain:

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & -4 & -3/2 \\ 1 & 0 & 1 \end{pmatrix}$$

We set the next goal: obtaining a zero at the first column of row 3 by using the first element of row 1:

$$\begin{pmatrix} \boxed{2} & 4 & 1 \\ 0 & -4 & -3/2 \\ \boxed{1} & 0 & 1 \end{pmatrix}$$

Now the operation is $H_{31}\left(\frac{-1}{2}\right)$. We obtain

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & -4 & -3/2 \\ 0 & -2 & 1/2 \end{pmatrix}$$

Once the first column is simplified, we go to the second one. We want to use the element at row 2, column 2 to obtain a zero at row 3, column 2:

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & \boxed{-4} & -3/2 \\ 0 & \boxed{-2} & 1/2 \end{pmatrix}$$

The operation is $H_{32}\left(\frac{-(-2)}{-4}\right)$, that is, $H_{32}\left(\frac{1}{-2}\right)$. Finally, we obtain:

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & -4 & -3/2 \\ 0 & 0 & 5/4 \end{pmatrix}$$

## 7.3 Simplifying the process.

In the general formula that we have just seen, $H_{ji}\left(\frac{-q}{p}\right)$, the simplest situation appears when $p = 1$. In this case, the denominator is 1 and the operations are simpler. Sometimes we can obtain a pivot 1 by changing the row order.

We can apply this idea in the matrix of the previous example. We see that there is a 1 in the first column.

$$A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 2 & 0 \\ \boxed{1} & 0 & 1 \end{pmatrix}$$

So we can start by moving this element to the first row. We swap rows 1 and 3 by using the elementary operation $H_{13}$:

$$\begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \\ 2 & 4 & 1 \end{pmatrix}$$

Now we apply the general method: we want to obtain a zero element at row 2, column 1 with the element of row 1, column 1. To get this we apply the operation $H_{21}\left(\frac{-3}{1}\right)$, which simplifies to $H_{21}(-3)$:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -3 \\ 2 & 4 & 1 \end{pmatrix}$$

Then, we get a zero on the third row, by $H_{21}\left(\frac{-2}{1}\right) = H_{21}(-2)$:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -3 \\ 0 & 4 & -1 \end{pmatrix}$$

Finally, we would continue with the same idea to get zeros under the diagonal at the second column.

## 7.4 Example of congruence diagonalization.

The same formula used in the row reduction is applied to diagonalize a matrix by congruence. Suppose we have the matrix

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 8 & 0 \\ 6 & 0 & 20 \end{pmatrix}$$

We want to obtain a zero at row 2, column 1 by using the element at row 1, column 1:

$$\begin{pmatrix} \boxed{2} & 4 & 6 \\ \boxed{4} & 8 & 0 \\ 6 & 0 & 20 \end{pmatrix}$$

The operation is $H_{21}\left(\frac{-4}{2}\right)$ which simplifies to $H_{21}(-2)$:

$$\begin{pmatrix} 2 & 4 & 6 \\ 0 & 0 & -12 \\ 6 & 0 & 20 \end{pmatrix}$$

Since we are doing congruence we must do next the same column operation $V_{21}(-2)$:

$$\begin{pmatrix} 2 & 0 & 6 \\ 0 & 0 & -12 \\ 6 & -12 & 20 \end{pmatrix}$$

Next step: obtain a zero in the third row.

$$\begin{pmatrix} \boxed{2} & 0 & 6 \\ 0 & 0 & -12 \\ \boxed{6} & -12 & 20 \end{pmatrix}$$

The operation is $H_{31}\left(\frac{-6}{2}\right) = H_{31}(-3)$:

$$\begin{pmatrix} 2 & 0 & 6 \\ 0 & 0 & -12 \\ 0 & -12 & 2 \end{pmatrix}$$

Again, we do the same operation for columns $V_{31}(-3)$.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -12 \\ 0 & -12 & 2 \end{pmatrix}$$

Next we should obtain zeros on the second column by using the second element of the diagonal. But:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & \boxed{0} & -12 \\ 0 & -12 & \boxed{2} \end{pmatrix} \qquad 0 \text{ is not valid as a pivot!.}$$

To solve this problem, we look for nonzero elements in the remaining of the diagonal. We find one in the third row. Thus, we swap rows 2 and 3 and columns 2 and 3:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -12 \\ 0 & -12 & 2 \end{pmatrix} \xrightarrow{H_{23}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -12 & 2 \\ 0 & 0 & -12 \end{pmatrix} \xrightarrow{V_{23}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -12 \\ 0 & -12 & 0 \end{pmatrix}$$

Now we can use the second element of the diagonal to obtain zeros below it:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & \boxed{2} & -12 \\ 0 & \boxed{-12} & 0 \end{pmatrix}$$

The operation is $H_{32}\left(\frac{-(-12)}{2}\right) = H_{32}(6)$:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -12 \\ 0 & 0 & -72 \end{pmatrix}$$

and we do the same operation for columns $V_{32}(6)$:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -72 \end{pmatrix}$$