2. Determinants.

1 Basic notions about permutations.

Definition 1.1 *Given a natural number n the* **set of permutations of n elements** *is the set of possible rearrangements of the integer numbers $1, 2, \dots, n$. It is denoted by $Perm(n)$.*

The elements of $Perm(n)$ are called **permutations.**

From this, an element of $Perm(n)$ corresponds to a bijective map:

$$\begin{array}{rccc} \sigma: & \{1,\dots,n\} & \longrightarrow & \{1,\dots,n\} \\ & 1 & \longrightarrow & \sigma(1) \\ & \vdots & & \vdots \\ & n & \longrightarrow & \sigma(n) \end{array}$$

$\sigma(i)$ indicates which number we place at the ith position.

We know that n elements can be ordered in $n!$ different ways, so the set $Perm(n)$ has $n!$ elements.

Given a permutation σ we can consider the **inverse permutation** σ^{-1} which corresponds to the inverse funcion of σ.

Definition 1.2 *A* **transposition** *is a permutation that keeps all the elements in the same order except that two of them are swapped.*

It can be proved that every permutation may be represented as a composition of transpositions. This allows us to define the following:

Definition 1.3 *Given a permutation σ we call* **signature** *of σ and denote by $\epsilon(\sigma)$ the number:*

$$\epsilon(\sigma) = (-1)^k$$

where k is the number of transpositions in the decomposition of σ.

A permutation can be decomposed into transpositions in different ways. However, the parity of the number of transpositions in each decomposition is the same. Thus, if a permutation is a composition of an even number of transpositions, the signature will be $+1$; on the contrary, if it is a composition of an odd number of transpositions, the signature will be -1.

On the other hand, given a decomposition of a permutation into transpositions, it is clear that the inverse permutation σ^{-1} is obtained by composing the inverse of the transpositions. From this, the signature of a permutation and the signature of its inverse coincide:

$$\epsilon(\sigma) = \epsilon(\sigma^{-1}).$$

2 Determinant of a square matrix.

2.1 Definition.

Given a square matrix A, we will denote its ith row by A_i:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}; \qquad A_i = (a_{i1}, a_{i2}, \dots, a_{in}).$$

Definition 2.1 *The* **determinant** *of a square matrix A is a map from the set of square matrices over the field $\mathbb{K}$:*

$$\mathcal{M}_{n\times n} \longrightarrow \mathbb{K}; \quad |A| = det(A_1, A_2, \dots, A_n) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

satisfying the following properties for any $i, j \in \{1, 2, \dots, n\}$:

1. *It is multilinear:*

$$det(A_1, \dots, A_i + A_i', \dots A_n) = det(A_1, \dots, A_i, \dots A_n) + det(A_1, \dots, A_i', \dots A_n).$$

$$det(A_1, \dots, \alpha A_i, \dots A_n) = \alpha \cdot det(A_1, \dots, A_i, \dots A_n), \text{ for any } \alpha \in \mathbb{K}.$$

2. *It is antisymmetric:*

$$det(A_1, \dots, A_i, \dots, A_j, \dots, A_n) = -det(A_1, \dots, A_j, \dots, A_i, \dots, A_n).$$

3. $|I_n| = 1$.

2.2 Properties.

1. The determinant of any matrix with two equal rows is 0:

$$det(A_1, \dots, A_i, \dots, A_i, \dots, A_n) = 0.$$

Proof: As a consequence of the antisymmetry condition:

$$det(A_1, \dots, A_i, \dots, A_i, \dots, A_n) = -det(A_1, \dots, A_i, \dots, A_i, \dots, A_n)$$

so:

$$2det(A_1, \dots, A_i, \dots, A_i, \dots, A_n) = 0.$$

2. The determinant of any matrix with a null row is 0:

$$det(A_1, \dots, 0, \dots, A_n) = 0.$$

Proof. Since the determinant map is multilinear:

$$det(A_1, \dots, 0, \dots, A_n) = 0 \cdot det(A_1, \dots, 0, \dots, A_n) = 0.$$

3. If one of the rows of a matrix is multiplied by a scalar and then added to another row of the same matrix, the resulting matrix has the same determinant as the original one:

$$det(A_1, \dots, A_i, \dots, A_j + \lambda \cdot A_i, \dots, A_n) = det(A_1, \dots, A_i, \dots, A_j, \dots, A_n).$$

Proof: We use multilinearity of the determinant and the first property:

$$\begin{aligned} & det(A_1, \dots, A_i, \dots, A_j + \lambda \cdot A_i, \dots, A_n) = \\ & = det(A_1, \dots, A_i, \dots, A_j, \dots, A_n) + \lambda\, det(A_1, \dots, A_i, \dots, A_i, \dots, A_n) = \\ & = det(A_1, \dots, A_i, \dots, A_j, \dots, A_n). \end{aligned}$$

2.3 Computation of the determinant

Let A be an $n \times n$ matrix.

We denote by E_i the row with a 1 at the ith position and 0's at all the remaining ones.

$$E_i = (0, \dots, 0, \underbrace{1}_{i}, 0, \dots, 0).$$

With this notation:

$$A_i = (a_{i1}, a_{i2}, \dots, a_{in}) = \sum_{j=1}^{n} a_{ij} E_j.$$

Let us compute the determinant of A:

$$det(A) = det(A_1, A_2, \dots, A_n) = det(\sum_{i_1=1}^{n} a_{1i_1} E_{i_1}, \sum_{i_2=1}^{n} a_{2i_2} E_{i_2}, \dots, \sum_{i_n=1}^{n} a_{ni_n} E_{i_n})$$

Applying the multilinearity of the determinant, we obtain:

$$det(A) = \sum_{i_1, i_2, \dots, i_n = 1}^{n} a_{1i_1} a_{2i_2} \dots a_{ni_n} \cdot det(E_{i1}, E_{i2}, \dots, E_{in}).$$

When two repeated rows appear in the expression $det(E_{i1}, E_{i2}, \dots, E_{in})$ the corresponding determinant is zero. In other case, we can rearrange the rows as $det(E_1, E_2, \dots, E_n)$. For each change of position of two rows there is a change of sign. As a consequence of this and with the notation described in the preliminary section of the chapter, the formula of the determinant is:

$$\boxed{det(A) = \sum_{\sigma \in Perm(n)} \epsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}}$$

2.4 Determinant of the transpose of a matrix.

Theorem 2.2 *If A is any $n \times n$ matrix*

$$det(A) = det(A^t)$$

Proof: We will use the formula obtained in the previous section:

$$\begin{aligned} |A^t| & = \sum_{\sigma \in Perm(n)} \epsilon(\sigma) (A^t)_{1\sigma(1)} (A^t)_{2\sigma(2)} \dots (A^t)_{n\sigma(n)} = \\ & = \sum_{\sigma \in Perm(n)} \epsilon(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \dots a_{\sigma(n)n}. \end{aligned}$$

We now rewrite each monomial above by using the inverse permutation of each σ. Recall that the signature of a permutation and the signature of its inverse coincide. From this:

$$|A^t| = \sum_{\sigma \in Perm(n)} \epsilon(\sigma^{-1}) a_{1\sigma^{-1}(1)} a_{2\sigma^{-1}(2)} \dots a_{n\sigma^{-1}(n)}.$$

Finally, in the above summation σ runs through all the possible permutations of n elements, so σ^{-1} also runs through all the possible permutations of n elements. We obtain:

$$det(A^t) = \sum_{\rho \in Perm(n)} \epsilon(\rho) a_{1\rho(1)} a_{2\rho(2)} \dots a_{n\rho(n)} = det(A).$$

■

The main consequence of this theorem is:

Any property of the determinant which depends on the rows of the matrix is also true for columns.

2.5 Determinant of the product of two matrices.

Theorem 2.3 *Let A, B be two $n \times n$ matrices. Then*

$$det(AB) = det(A) det(B).$$

Proof: Denote by $C = AB$ the product matrix of A and B. We know that:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad \text{and from this} \quad C_i = \sum_{k=1}^{n} a_{ik} B_k.$$

Then:

$$\begin{aligned} |C| & = det(C_1, C_2, \dots, C_n) \\ & = det(\sum_{k_1=1}^{n} a_{1k_1} B_{k_1}, \sum_{k_2=1}^{n} a_{1k_2} B_{k_2}, \dots, \sum_{k_n=1}^{n} a_{1k_n} B_{k_n}) = \\ & = \sum_{k_1, k_2, \dots, k_n=1}^{n} a_{1k_1} a_{1k_2} \dots a_{1k_n} det(B_{k_1}, B_{k_2}, \dots, B_{k_n}). \end{aligned}$$

Whenever $k_i = k_j$ for some i and j, the determinant $det(B_{k_1}, B_{k_2}, \ldots, B_{k_n})$ is zero. Thus, we consider only the terms where the indices $k_1, \ldots, k_n$ take all possible values between 1 and n. That is, these indices define a permutation. Therefore the previous expression can be written as:

$$|C| = \sum_{\sigma \in Perm(n)} a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)} det(B_{\sigma(1)}, B_{\sigma(2)}, \ldots, B_{\sigma(n)})$$

We can rearrange the rows $B_{\sigma(1)}, B_{\sigma(2)}, \ldots, B_{\sigma(n)}$ as $B_1, \ldots, B_n$. For each change os position of two rows there is a change of sign. We obtain:

$$|C| = \sum_{\sigma \in Perm(n)} \epsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)} det(B_1, B_2, \ldots, B_n) = det(A) det(B).$$

3 Cofactor expansion of the determinant.

3.1 Minors of a matrix.

Definition 3.1 *A* **minor of order** r *of a matrix* A *is the determinant of some* $r \times r$ *matrix, obtained from* A *by removing some of its rows and columns:*

$$A \begin{pmatrix} i_1 i_2 \ldots i_r \\ j_1 j_2 \ldots j_r \end{pmatrix} = \begin{vmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \ldots & a_{i_1 j_r} \\ a_{i_2 j_1} & a_{i_2 j_2} & \ldots & a_{i_2 j_r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_r j_1} & a_{i_r j_2} & \ldots & a_{i_r j_r} \end{vmatrix}$$

with $i_1 < i_2 < \ldots < i_r$ *and* $j_1 < j_2 < \ldots < j_r$.

3.2 Cofactors of a matrix.

Definition 3.2 *Given a square matrix* $A \in \mathcal{M}_{n \times n}$ *the* **cofactor** A_{ij} *is the minor obtained by suppressing the* i*-th row and the* j*-th column, multiplied by the factor* $(-1)^{i+j}$..

The following properties hold:

Proposition 3.3 *Given a square matrix* $A \in \mathcal{M}_{n \times n}$*:*

$$A_{11} = \begin{vmatrix} 1 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

Proof: Let us denote by B the matrix:

$$B = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

Applying the formula for the determinant, we have:

$$|B| = \sum_{\sigma \in Perm(n)} \epsilon(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \ldots b_{n\sigma(n)}$$

But $b_{ij} = 0$ when $i = 1$ and $j > 1$. Moreover $b_{11} = 1$. If σ is a permutation that changes the position of 1 ($\sigma(1) \neq 1$) then $b_{1\sigma(1)} = 0$ and the corresponding term of the sum is zero. From this, we can consider only permutations that leave 1 fixed. These correspond to permutations on the set $\{2, \ldots, n\}$, so:

$$\begin{aligned} |B| &= \textstyle\sum_{\sigma \in Perm(2,\ldots,n)} \epsilon(\sigma) b_{2\sigma(2)} b_{3\sigma(3)} \ldots b_{n\sigma(n)} \\ &= \textstyle\sum_{\sigma \in Perm(2,\ldots,n)} \epsilon(\sigma) a_{2\sigma(2)} a_{3\sigma(3)} \ldots a_{n\sigma(n)} = A_{11}. \end{aligned}$$

■

Corollary 3.4 *Given a square matrix* $A \in \mathcal{M}_{n \times n}$*:*

$$A_{ij} = \begin{vmatrix} a_{11} & \ldots & a_{1j} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i-1\,1} & \ldots & a_{i-1\,j} & \ldots & a_{i-1\,n} \\ 0 & \ldots & 1 & \ldots & 0 \\ a_{i+1\,1} & \ldots & a_{i+1\,j} & \ldots & a_{i+1\,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nj} & \ldots & a_{nn} \end{vmatrix}$$

Proof: It is sufficient to apply the previous result. We can move the i-th row and the j-th column to the position $1, 1$. We make $i - 1, j - 1$ sign changes, so we have to multiply by the factor:

$$(-1)^{i-1+j-1} = (-1)^{i+j}.$$

■

3.3 Expansion of the determinant along a row or a column.

Let us see how to calculate the determinant of a square matrix A by **expanding along the ith row**:

$$\begin{aligned}|A| &= det(A_1,\dots,A_i,\dots,A_n) = \\ &= det(A_1,\dots,a_{i1}E_1\dots + a_{in}E_n,\dots,A_n) = \\ &= a_{i1}det(A_1,\dots,E_1,\dots,A_n) + \dots + a_{in}det(A_1,\dots,E_n,\dots,A_n) \\ &= a_{i1}A_{i1} + \dots + a_{in}A_{in}.\end{aligned}$$

We deduce the following formula:

$$\boxed{|A| = \sum_{j=1}^{n} a_{ij}A_{ij}}$$

Analogously the formula to calculate the determinant **expanding along the j-th column** is:

$$\boxed{|A| = \sum_{i=1}^{n} a_{ij}A_{ij}}$$

The importance of both formulas is that they allow us to reduce the calculation of an $n\times n$ determinant to that of an $(n-1)\times(n-1)$ one. We can successively apply this reduction of the dimension of the problem as many times as we wish.

4 Rank of a matrix.

Definition 4.1 *Given a square matrix A we define the* **rank of** *A as the order of a highest-order nonvanishing minor of the matrix.*

As a consequence of the properties of the determinant we have:

1. The rank of a matrix coincides with the rank of its transpose.
2. Elementary row or column operations does not change the rank.

5 Inverse of a matrix.

The use of determinants provides a method to calculate the inverse of a square matrix A. Note that a necessary condition for a square matrix A to have an inverse is that its determinant if not null:

$$A\cdot A^{-1} = Id \quad\Rightarrow\quad det(A)det(A^{-1}) = 1 \quad\Rightarrow\quad det(A)\neq 0 \text{ and } det(A^{-1}) = \frac{1}{det(A)}.$$

We introduce the following definition:

Definition 5.1 *Given a square matrix A we call* **adjoint matrix** *of A the transpose of the matrix of cofactors:*

$$(adjA) = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn}\end{pmatrix}.$$

Let us now calculate the product:

$$B = A\cdot(adjA).$$

We have:

$$b_{ij} = \sum_{k=1}^{n} a_{ik}(adjA)_{kj} = \sum_{k=1}^{n} a_{ik}A_{jk}.$$

If $i=j$:

$$b_{ii} = \sum_{k=1}^{n} a_{ik}A_{ik} = |A|.$$

If $i\neq j$,

$$b_{ij} = \sum_{k=1}^{n} a_{ik}A_{jk} = \sum_{k=1}^{n} a_{ik}C_{ik} = |C|,$$

where C is matrix equal to A, except in the j-th row that is equal to the i-th. From this $|C| = 0$ and it remains:

$$A\cdot(adjA) = |A|Id.$$

On the other hand:

$$(adjA)\cdot A = (A^t\cdot(adjA)^t)^t = (A^t\cdot(adjA^t))^t = |A^t|Id^t = |A|Id$$

Threfore **if** $|A|\neq 0$:

$$\left.\begin{aligned} A\cdot\frac{1}{|A|}(adjA) = Id \\ \frac{1}{|A|}(adjA)\cdot A = Id\end{aligned}\right\} \quad\Rightarrow\quad A^{-1} = \frac{1}{|A|}(adjA).$$

We have proved the following theorem:

Theorem 5.2 *Let A be an n-dimensional square matrix. A is invertible if and only if its determinant is nonzero. In this case:*

$$\boxed{A^{-1} = \frac{1}{|A|}(adjA)} \qquad \textit{and} \qquad \boxed{|A^{-1}| = \frac{1}{|A|}}$$