LINEAR ALGEBRA I — Problem Set 7

Endomorphisms — (Academic year 2022–2023)

1.– Given the matrix:

$$A = \begin{pmatrix} 3 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

(a) Determine whether A is triangularizable .

(b) Find the eigenvalues and eigenvectors of A.

2.– Given the matrix:

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

(i) Determine whether A is diagonalizable.

(ii) Find the eigenvectors of A.

(iii) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(iv) Compute $trace(A^{20})$ and $det(A^{20})$.

(Final exam, July 2021)

3.– Let be the endomorphism of $\mathbb{R}^3$,

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \qquad f(x, y, z) = (x + y + 3z, 4y, 3x - y + z)$$

(i) Find the matrix F_C associated to f relative to the canonical basis.

(ii) Prove that f is diagonalizable.

(iii) Find the eigenvectors of f.

(iv) Find a basis B of $\mathbb{R}^3$ such that F_B is diagonal.

(v) Is there any $n \in \mathbb{N}$ such that $trace(F_C^n) = 3^{2017}$?

(Final exam, January 2018)

4.– For each real number $a \in \mathbb{R}$ the endomorphism of $\mathbb{R}^3$ is defined:

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad f(x, y, z) = (x + 2y - 2z, 2x + y - 2z, az)$$

(i) Determine in terms of a when the endomorphism f is diagonalizable and/or triangularizable.

(ii) For $a = 0$ find a basis B such that F_B is diagonal.

(iii) For $a = 1$ obtain $trace(F_C^{1515})$.

(Final exam, January 2016)

5.– Let be the matrix $A = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$:

(i) Find its eigenvalues and eigenvectors.

(ii) Find an invertible matrix P such that $P^{-1}AP = D$ is diagonal.

(iii) Obtain A^n in terms on $n \in \mathbb{N}$.

(Final exam, January 2022)

6.– Given $a \in \mathbb{R}$ consider the endomorphism of $\mathbb{R}^3$,

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \qquad f(x, y, z) = (x + y + z, x + y + az, x + y + z)$$

(i) Find the associated matrix F_C to f relative to the canonical basis.

(ii) Determine in terms of a when the endomorphism f is diagonalizable and/or triangularizable.

(iii) For $a = -2$ calculate the eigenvectors of f.

(iv) For $a = 1$ find a basis of B such that F_B is diagonal.

(v) For $a = 5$ obtain the trace of F_C^{11}.

(Final exam, January 2020)

7.– Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & b \\ a & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 2 \end{pmatrix}$$

(i) Determine in terms of a and b when A is diagonalizable and/or triangularizable by similarity.

(ii) For the values of a and b for which the matrix is diagonalizable:

(ii.a) Find the eigenvectors of A.

(ii.b) Find a diagonal matrix D and a matrix P such that $D = P^{-1}AP$.

(ii.c) Find $trace(A^{40})$.

(Final exam, July 2017)

8.– Given the numbers $a, b \in \mathbb{R}$ we define the matrix:

$$A = \begin{pmatrix} a & -a & 0 \\ 1 & -1 & 0 \\ b & 1 & 0 \end{pmatrix}$$

(i) Determine in terms of a and b when A is diagonalizable and/or triangularizable by similarity. In the cases for which it is diagonalizable, obtain a diagonal matrix similar to A.

(ii) For $a = 2$ and $b = -1$ obtain A^{2017}.

(iii) For $a = 0$ and $b = 1$ find its eigenvectors.

(Final exam, January 2017)

9.– Given $a \in \mathbb{R}$ consider the matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & a \\ 2 & 1 & -2 & 0 \\ 1 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(i) Determine the values of a for which A is diagonalizable by similarity.

(ii) Find a such that $trace(A^4) = 19$.

(iii) For $a = 0$ obtain the eigenvectors of A. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(Final exam, July 2018)

10.– Given $a \in \mathbb{R}$ consider the matrix:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ a & 3 & 1 & 0 \\ 4 & 1 & 1 & 0 \end{pmatrix}.$$

(i) Determine the values of a for which A is diagonalizable by similarity.

(ii) For $a = 0$ find the eigenvectors of A.

(iii) For the values of a for which it is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(Final exam, January 2021)

11.– Given the matrix $A = \begin{pmatrix} 1 & 1 \\ a & 1 \end{pmatrix}$ discuss for which values of a is diagonalizable and/or triangularizable by similarity. In the cases where it is diagonalizable, give the corresponding diagonal form.

(Final exam, January 2018)

12.– Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an endomorphism. If $(1,2) \in ker(f)$ and $(0,1)$ is an eigenvector associated to 2, obtain $f(1,5)$. Is f diagonalizable?

(Final exam, January 2021)

13.– Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map such that

i) $Ker(f) = \{(x,y,z) \in \mathbb{R}^3 | x - y = 0, \ x - z = 0\}$

ii) The vectors $(1,1,0)$ and $(1,0,0)$ are eigenvectors associated with the same eigenvalue.

iii) $trace(F_C) = 4$.

Find the associated matrix to f relative to the canonical basis.

((Final exam, January 2017)

14.– Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an endomorphism. Given that $(1,2), (0,1)$ are eigenvectors of f associated respectively to 1 and 2, obtain $f(2,0)$. If f diagonalizable?

(Exam July, 2017)

15.– Let $C = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$ be the canonical basis of $\mathbb{R}^4$ and let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be an endomorphism such that

$$f(\vec{e}_1) = \vec{e}_1 + \vec{e}_3$$
$$ker(f) = \mathcal{L}\{\vec{e}_1 + \vec{e}_3,\ \vec{e}_2 - \vec{e}_4\}$$
$\vec{e}_2 + \vec{e}_3$ is an eigenvector of f.
The only eigenvalues of f are 0 and 2.

(i) Find the matrix associated to f relative to the canonical basis.

(ii) Find all eigenvalues, their geometric multiplicities and their associated eigenvectors.

(Exam, January 2014)

16.– In $\mathbb{R}^3$ consider two complementary subspaces U and V with $dim(U) = 2$. Let $p : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the projection map over U along V and P_C the associated matrix with respect to the canonical basis.

(i) Is P_C diagonalizable by similarity?

(ii) Find the eigenvalues of P_C and their algebraic and geometric multiplicities.

(iii) If we further know that $p(1, 0, 1) = (0, 0, 0)$, find the implicit equations of V with respect to the canonical base .

(iv) If we further know that $p(1, 2, 3) = (-1, 2, 1)$ find $p(-1, 2, 1)$ and the projection of $(1, 2, 3)$ over V along U.

(Final exam, June 2014)

17.– Give examples of matrices (if they exist) satisfying each one of the following conditions, and justify in each case that the proposed matrix meets the requirements.

(i) A matrix $A \in M_{2\times2}(\mathbb{R})$ which is triangularizable by similarity but not diagonalizable.

(ii) A matrix $A \in M_{2\times2}(\mathbb{R})$ which is not triangularizable by similarity.

(iii) A matrix $A \in M_{2\times2}(\mathbb{R})$ which is diagonalizable by similarity but not triangularizable.

(Final exam, January 2016)

18.– Consider the linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ which satisfies

(i) $(1, 1)$, $(1, 2)$ are eigenvectors of f.

(ii) $f(2, 2) \neq (0, 0)$.

(iii) $trace(F_C) = 2$.

(iv) $dim(Im(f)) = 1$.

Find the matrix associated to f with respect to the canonical basis.

(Final exam, January 2019)

19.– Determine and argument whether the following statements are true or false:

(i) A matrix $A \in M_{4\times4}(\mathbb{R})$ with four different real eigenvalues is always diagonalizable by similarity.

(ii) Any square matrix which diagonalizable by similarity is also diagonalizable by congruence.

(iii) The sum of two matrices which are both diagonalizable by similarity is diagonalizable by similarity.

(iv) If 0 is an eigenvalue of the endomorphism f then f is not injective.

(Exam, January 2015)

I.– Let A be a square matrix whose characteristic polynomial is $p(\lambda) = \lambda^2 - 1$. Prove that $A^2 - Id = \Omega$.

(Final exam, June 2007)

II.– On a vector space E of dimension n, we consider two complementary vector subspaces U and V, with $dim(U) = k$. Consider the projection endomorphism over U along V:

$$p : E \longrightarrow E$$

(i) Justify that p is diagonalizable.

(ii) Find the characteristic polynomial of p, its eigenvalues and the corresponding characteristic subspaces.

(Exam, January 2012)

III.– Let $T \in M_{n\times n}(\mathbb{R})$ be a matrix such that the sum of the elements of each of its rows is 2011.

(i) Prove that 2011 is an eigenvalue of T.

(ii) Find a vector $u \in \mathbb{R}^n$ such that $Tu = 2011u$

(iii) Prove that u is also an eigenvector of T^m, for any natural m.

(iv) Prove that the sum of the elements of each of the rows of T^m is a constant K_m and obtain its value in terms of m.

(Exam June, 2011)

IV.– Given the matrix:

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 & -1 \\ 0 & 2 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

(a) Compute its eigenvalues and eigenvectors.

(b) Is it diagonalizable and/or triangularizable by similarity?.

(c) Find, if possible, a Jordan form J and an invertible matrix P such that $J = P^{-1}AP$.

(Exam, December 2009)

V.– Consider the matrix

$$A = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

(b) Find the Jordan form J of A and an invertible P such that $J = P^{-1}AP$.

(d) Obtain A^{10}.

(First partial, January 2006)

VI.– Given the matrix

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}.$$

a) Find the eigenvalues and eigenvectors of A.

b) Compute a Jordan form associated to A, giving the corresponding transition matrix P.

c) Calculate $(A + Id)^{2010}$.

(Exam, September 2010)

VII.– Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & a \\ 0 & 1 & 0 \\ 0 & -a & a+1 \end{pmatrix}, \qquad a \in \mathbb{R}.$$

(a) Determine the values of a for which the matrix is triangularizable and those for which it is diagonalizable.

(b) For the values of a for which A is triangularizable but not diagonalizable, compute the Jordan canonical form J of A and a matrix P such that $J = P^{-1}AP$.

(c) For $a = -1$, compute A^n for any $n \in \mathbb{N}$.

(Final exam, June 2005)

VIII.– Given the matrix

$$A = \begin{pmatrix} -a & 0 & -1 & a-1 \\ a & 1 & 0 & -a \\ 1+2a & 0 & 2 & 1-2a \\ -1-a & 0 & -1 & a \end{pmatrix}$$

where a is a real parameter:

a) Discuss in terms of $a \in \mathbb{R}$ when A is triangularizable and/or diagonalizable (by similarity).

b) For $a = 1$, find a Jordan form associated to A and the corresponding transition matrix.

(Exam, September 2007)

IX.– Let A be the matrix:

$$A = \begin{pmatrix} -3 & -2 & 0 & a \\ 4 & 3 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 4 & 3 \end{pmatrix}$$

a) Determine the values of a for which the matrix is triangularizable. Indicate when it is also diagonalizable.

b) When possible, calculate the corresponding diagonal or Jordan matrix in terms of the parameter a.

c) For $a = 0$ compute the eigenvectors of A. Also find an invertible matrix P such that $J = P^{-1}AP$ where J is the Jordan form.

(Exam, December 2007)

X.– Let A be a square matrix. Suppose that $(\lambda - 2)^{12}$ is the characteristic polynomial of A:

$$\text{geometric multiplicity}(2) = 6, \quad rank((A - 2Id)^2) = 2, \quad dim(ker((A - 2Id)^3) = 11.$$

Is A triangularizable by similarity? If so, compute a Jordan matrix similar to A.

(Exam, January 2010)

XI.– Find a matrix A satisfying the following conditions:

(1) $\dim(\text{Ker}A) = 1$

(2) $\lambda = 1$ is an eigenvalue with algebraic multiplicity 4.

(3) $\lambda = 2$ is an eigenvalue with algebraic multiplicity 3.

(4) $\text{rg}(A - I) = 6$, $\text{rg}(A - I)^2 = 4$

(5) $\text{rg}(A - 2I) = 7$, $\text{rg}(A - 2I)^3 = 5$

(Final exam, January 2003)

XII.– Find a 7×7 real matrix A, such that

$$\text{rank}(A) = 4, \quad A^4 = \Omega, \quad A^3 \neq \Omega.$$

(First partial exam, February 2000)

XIII.–

Find a 6×6 real matrix A satisfying the following conditions:

$$\dim \text{Ker}A = 2, \quad \dim \text{Ker}A^2 = 4, \quad \text{rank}(A - I) = 4.$$

(Exam, September 1999)

XIV.– Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the endomorphism given by

$$f(x, y, z) = ((\alpha - 1)x + \alpha y + (\alpha - 2)z,\ x + y + z,\ x - \alpha y + 2z)$$

(a) Find the values of α for which f is a diagonalizable endomorphism; for those values, find a basis of characteristic vectors.

(b) Find the value of α for which f has a triple eigenvalue. Find the Jordan canonical form and an associated basis.

(Exam, September 2004)

XV.– Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all polynomials of degree less than or equal to 2. Consider the map

$$f : \mathcal{P}_2(x) \longrightarrow \mathcal{P}_2(x); \qquad f(p(x)) = p(x) - p'(x)$$

(a) Show that f is an endomorphism.

(b) Obtain the associated matrix to f with respect to the canonical basis of $\mathcal{P}_2(x)$.

(c) Compute its eigenvalues and its eigenvectors.

(d) If f is triangularizable, compute its Jordan form and the basis relative to which it is expressed.

(e) Describe, if possible, the inverse mapping of f.

(Exam, January 2005)

XVI.– Let $A \in M_{2\times 2}(\mathbb{R})$. It is known that A^2 is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ and $trace(A) = 1$.

a) Find the eigenvalues of A.

b) Is A diagonalizable by similarity?

(Exam, September 2010)
