LINEAR ALGEBRA I — Problem Set 6

Linear maps (Academic year 2022–2023)

1.– Determine which of the following maps are homomorphism, monomorphisms, epimorphisms or isomorphisms. Obtain also the matrix representation with respect to the canonical basis, the kernel (basis and equations) and the image subspace.

(a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 3x + 2$

(b) $g : \mathbb{R}^2 \to \mathbb{R}^3, \quad g(x, y) = (x,\ y,\ x + y)$

(c) $h : \mathbb{R}^2 \to \mathbb{R}^2, \quad h(x, y) = (xy,\ x - 2y)$

(d) $u : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R}), \quad u(p(x)) = p'(x)$

(e) $v : \mathcal{M}_{2\times 3} \to \mathcal{S}_3, \quad v\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} a+b & a-b & c \\ a-b & d & e+f \\ c & e+f & e-f \end{pmatrix}$

2.– Consider the linear map:

$$f : \mathbb{R}^3 \to \mathbb{R}^2, \qquad f(x, y, z) = (x + y + z, x - y)$$

(i) Find the matrix associated to f with respect to the canonical bases of $\mathbb{R}^3$ and $\mathbb{R}^2$.

(ii) Prove that $B = \{(1, 0, 1), (1, 1, 0), (1, 0, 0)\}$ and $B' = \{(1, 2), (1, 0)\}$ are bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

(iii) Find the implicit equations of $ker(f)$ with respect to the basis B.

(iv) Find the matrix associated to f with respect to the bases B and B'.

(Final exam, July 2018)

3.– Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all polynomials of degree less than or equal to 2 and let $S_2(\mathbb{R})$ be the vector space of real symmetric matrices of order 2. Define:

$$f : \mathcal{P}_2(\mathbb{R}) \to S_2(\mathbb{R}), \qquad f(p(x)) = \begin{pmatrix} p(0) & p(1) \\ p(1) & p(2) \end{pmatrix}$$

(i) Prove that f is a linear map.

(ii) Find the matrix associated to f with respect to the canonical basis.

(iii) Find the matrix associated to f with respect to the basis $B = \{1, (x - 1), (x - 2)^2\}$ of $\mathcal{P}_2(\mathbb{R})$ and $B' = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ of $S_2(\mathbb{R})$.

(Final exam, July 2016)

4.– In $\mathbb{R}^3$ consider the subspaces with the following implicit equations with respect to the canonical basis:

$$x_1 + 2x_2 + x_3 = 0; \qquad \begin{cases} x_1 & = & 2x_2 \\ x_2 & = & -x_3 \end{cases}$$

Find the matrix associated to the projection map over the first subspace along the second one, and vice versa.

5.– Given the map:

$$f : \mathcal{M}_{2\times 2}(\mathbb{R}) \longrightarrow \mathcal{M}_{2\times 2}(\mathbb{R}), \qquad f(A) = A - A^t$$

(i) Prove that f is a linear map.

(ii) Find the matrix associated to f with respect to the canonical basis.

(iii) Find the matrix associated to f with respect to the basis:

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

(iv) Find the parametric equations of $ker(f)$ and the implicit equations of $Im(f)$ with respect to the basis B.

(v) Give a set of matrices which is a basis of $ker(f)$.

(Final exam, June 2014)

6.– Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all polynomials of degree less than or equal to 2. Consider the map:

$$f : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R}), \quad f(p(x)) = x \cdot p'(x)$$

(i) Prove that f is a linear map.

(ii) Find the matrix associated to f with respect to the canonical basis $C = \{1, x, x^2\}$ of $\mathcal{P}_2(\mathbb{R})$.

(iii) Prove that the vectors $B = \{1, x-1, (x-1)^2\}$ are a basis of $\mathcal{P}_2(\mathbb{R})$.

(iv) Find the matrix associated to f with respect to the basis B.

(v) Find the parametric equations of $Im(f)$ with respect to the canonical basis and the implicit equations of $ker(f)$ with respect to the basis B.

(Final exam, January 2021)

7.– From a linear map $f : \mathbb{R}^3 \to M_{2\times 2}(\mathbb{R})$ it is known that:

$$f(1, -2, 1) = Id, \quad ker(f) = \{(x, y, z) \in \mathbb{R}^3 | y - z = 0\}$$

Find the matrix associated to f with respect to the canonical bases of $\mathbb{R}^3$ and $M_{2\times 2}(\mathbb{R})$

(Final exam, January 2022)

8.– Consider the following map:

$$f : M_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{R}^2, \qquad f(A) = (trace(A), trace(AB)), \qquad \text{with } B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) Prove that it is a linear map.

(ii) Find the matrix associated to $f$with respect to the canonical bases of $M_{2\times 2}(\mathbb{R})$ and $\mathbb{R}^2$.

(iii) Find the parametric and implicit equations of $ker(f)$ with respect to the canonical basis of $M_{2\times 2}(\mathbb{R})$.

(iv) Find the matrix associated to f with respect to the canonical bases of B and B', where:

$$B = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\}, \qquad B' = \{(0,1), (1,1)\}.$$

(Final exam, January 2015)

9.– Find the associated matrix of an endomorphism f of $\mathbb{R}^4$ satisfying $f \circ f = 0$, $dim(Im(f)) = 2$, $(1,0,0,2) \in Im(f)$, $(0,1,1,0) \in ker(f)$.

(Final exam, July 2015)

10.– Given the linear maps:

$$f : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^2, \qquad f(p(x)) = (p(1), p(-1))$$
$$g : \mathbb{R}^2 \to \mathbb{R}^3, \qquad g(x,y) = (x+y, x-y, 2x-3y)$$

find the associated matrix of $h = g \circ f$ with respect to the canonical bases. Compute $h((x+1)^2)$.

(Final exam, July 2019)

11.– In the vector space $\mathcal{P}_2(\mathbb{R})$ consider the following subspaces:

$$U = \{p(x) \in \mathcal{P}_2(\mathbb{R}) | p(1) = 0\}, \quad V = \mathcal{L}\{x^2\}.$$

(i) Prove that they are complementary subspaces.

(ii) Find the associated matrix of the projection map over U along V with respect to the basis $C = \{1, x, x^2\}$.

(iii) Find the polynomial which is the projection of $2 - x + 3x^2$ over U along V.

(Final exam, January 2017)

12.– In be the vector space of all polynomials of degree less than or equal to 2, consider the subspaces:

$$U = \{p(x) \in \mathcal{P}_2(\mathbb{R}) | p(1) = 0\}, \qquad V = \{p(x) \in \mathcal{P}_2(\mathbb{R}) | p'(x) = 0\}$$

(i) Prove that U and V are complementary subspaces.

(ii) Find the associated matrix of the projection map over U along V with respect to the canonical basis.

(iii) Find a polynomial $p(x)$ such that its projection over U along V is $(x-1)^2$ and $p(0) = 0$.

(Final exam, July 2015)

13.– In $\mathbb{R}^3$ the projection map over a subspace U along V is:

$$p : \mathbb{R}^3 \to \mathbb{R}^3, \qquad p(x, y, z) = (-y + z, y, z)$$

(i) Find the parametric and implicit equations of U and V with respect to the canonical basis.

(ii) Find the projection of $\vec{w} = (2, 3, 1)$ over V along U.

(Final exam, January 2019)

14.– Determine and argument whether the following statements are true or false:

(i) In a vector space of dimension 10, a set of 11 vectors is a spanning set.

(ii) A linear map takes linearly dependent vectors into linearly dependent vectors.

(iii) A linear map $f : \mathbb{R}^5 \to \mathbb{R}^4$ can be injective.

(iv) A linear map $f : \mathbb{R}^5 \to \mathbb{R}^4$ is always surjective.

(Final exam, July 2016)

15.– Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an endomorphism satisfying $Im(f) = ker(f)$ and $f(1, 0) = (1, 1)$. Find $f(2, 3)$.

(Final exam, July 2019)

I.– Let $f : S_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear map, where $S_2(\mathbb{R})$ is the real vector space of all symmetric matrices of order 2. We know that

$$f\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \qquad f\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}, \qquad f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -3 & 3 \end{pmatrix}$$

Find:

(a) the matrix associated to f, indicating the bases in which is defined.

(b) the parametric equations of $Im(f)$, with respect to the basis $\left\{\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right\}$.

(c) the implicit equations of $ker(f)$, with respect to the basis $\left\{\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right\}$.

(d) Two 2-dimensional subspaces of S_2 and $M_{2\times 2}$, such that the restriction of f to them is bijective.

(First partial exam, January 2002)

II.– Let V be a real vector space of dimension 3. Let U and W be two complementary subspaces of V of dimensions 2 and 1 respectively. We call $f : V \to V$ the projection map over U along W. Let B be a basis of V.

Show that the matrix associated to f with respect to the basis B satisfies:

$$F_{BB}{}^n = F_{BB} \text{ for any } n \geq 1.$$

(First partial, January 2006)

III.– Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all polynomials of degree less than or equal to 2. Consider:

$$p(x) = 1 + x + x^2;\ q(x) = 1 + 2x^2;\ r(x) = x + x^2,$$

and

$$u = (2, 0, 1);\ v = (3, 1, 0);\ w = (1, -2, 3).$$

Given the linear map $f : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ defined as:

$$f(p(x)) = u;\ f(q(x)) = v;\ f(r(x)) = w.$$

(a) Find the associated matrix of f with respect to the canonical bases of V and $\mathbb{R}^3$.

(b) Find a basis B of V and a basis B' of $\mathbb{R}^3$ such that the matrix associated to f with respect to them is I_3.

(Exam, December 2005)

IV.– Let $f : \mathbb{R}^4 \longrightarrow \mathcal{P}_2(\mathbb{R})$ be a linear map defined as:

$$f(a,b,c,d) = (a+b)x^2 + bx + (c-d)$$

a) Find the matrix associated to f with respect to the bases B and B', where:

$$B = \{(1,1,0,0),(1,-1,0,0),(0,0,1,1),(0,0,1,-1)\}, \quad B' = \{1+x+x^2, 1+x, 1\}.$$

b) Find the parametric and implicit equations of $Ker(f)$ with respect to the canonical basis.

(First partial, June 2010)

V.– Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all polynomials of degree less than or equal to 2. Consider the following linear maps

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad f(x,y,z) = (x+y, y+z, x+z)$$
$$g : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}^3, \quad g(ax^2+bx+c) = (a-b, c+a-b, 2b-a)$$

and the bases $B = \{(1,1,0),(1,0,1),(0,1,1)\}$ and $C = \{1, x, x^2\}$.

Find the matrix associated to $f \circ g$ with respect to the bases C and B.

(Final exam, June 2009)

VI.– Let V be the space of real functions $f : \mathbb{R} \to \mathbb{R}$. Let $\phi : \mathbb{R}^3 \to V$ be the map which takes a triple (a,b,c) to the function $f_{(a,b,c)}$ defined as:

$$f_{(a,b,c)}(x) = a\text{sen}^2 x + b\cos^2 x + c, \qquad \forall x \in \mathbb{R}$$

a) Prove that ϕ is a linear map.

b) Find a basis of the kernel and a basis of the image. Determine whether ϕ is injective or surjective.

c) Verify that the set U formed by the constant functions is a vector subspace of V. Find its dimension and a basis.

d) Find $\phi^{-1}(U)$. If it is a subspace give a basis.

(First partial, February 1999)

VII.– Let $f : \mathbb{R}^5 \to \mathbb{R}^4$ and $g : \mathbb{R}^4 \to \mathbb{R}^5$ nonzero linear maps such that $g \circ f = 0$ and $dimIm(g) = 3$. Find $dimKer(f)$.

(Final exam, September 2007)

VIII.– Consider the linear map:

$$f : \mathbb{R}^3 \to \mathcal{M}_{2\times 2}(\mathbb{R}), \qquad f(x,y,z) = \begin{pmatrix} x+z & 2y+z \\ x+z & x-2y \end{pmatrix}$$

(i) Find the matrix associated to f with respect to the canonical bases of $\mathbb{R}^3$ and $\mathcal{M}_{2\times 2}(\mathbb{R})$.

(ii) Find the parametric equations of $ker(f)$ with respect to the canonical basis.

(iii) Prove that $B' = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ is a basis of $\mathcal{M}_{2\times 2}$.

(iv) If $B = \{(1,1,0),(1,1,1),(0,1,0)\}$ find the matrix associated to F with respect to the bases B and B'.

(Final exam, January 2018)

IX.–

(a) Decide if there is any linear map $f : \mathbb{R}^3 \to \mathbb{R}^4$ such that

$$\begin{aligned} \ker f &= \{(x^1,x^2,x^3) \in \mathbb{R}^3 : x^1 - x^3 = x^2 = 0\}, \\ \operatorname{Im} f &= \{(y^1,y^2,y^3,y^4) \in \mathbb{R}^4 : y^1 - y^2 = y^2 - y^3 = 0\}. \end{aligned}$$

If it exists, give the matrix (with respect to to the canonical bases of $\mathbb{R}^3$ and $\mathbb{R}^4$) of one that satisfies these conditions. If it does not exist, prove it.

(b) The same for

$$\begin{aligned} \ker f &= \{(x^1,x^2,x^3) \in \mathbb{R}^3 : 2x^1 - x^2 + x^3 = 0\}, \\ \operatorname{Im} f &= \{(y^1,y^2,y^3,y^4) \in \mathbb{R}^4 : y^1 + 2y^2 = y^1 - y^3 = 0\}. \end{aligned}$$

(First partial exam, February 2001)

X.– Let V a vector space of dimension $n \geq 1$ and let $f : V \longrightarrow V$ be an endomorphism. Determine and argument whether the following statements are true or false:

(a) If $f \circ f = id$ then $f = id$.

(b) If $f \circ f = 0$ then $f = 0$.

(c) $Ker(f - Id) \subset Im(f)$.

(Final exam, July 2012)
