LINEAR ALGEBRA I **Problem set 5**

Vector spaces (Academic year 2022–2023)

1.– In the real vector space $\mathbb{R}^2$ we consider the following subsets:

(a) $A = \{(x, y) \in \mathbb{R}^2 | \, x^2 + y^2 = 1\}$.

(b) $B = \{(x, y) \in \mathbb{R}^2 | \, x = 3y\}$.

(c) $C = \{(x, y) \in \mathbb{R}^2 | \, x + y = 1\}$

(d) $D = \{(x, y) \in \mathbb{R}^2 | \, x = y; \text{ e } y \geq 0\}$.

(e) $E = \{(x, y) \in \mathbb{R}^2 | \, x^2 + y^2 \geq 0\}$.

- Draw a picture of each one of them.
- Based on these pictures, deduce which of them are vector subspaces of $\mathbb{R}^2$.
- Prove it.

2.– In the vector space $M_{2\times2}(\mathbb{R})$ we consider the following sets of matrices:

$$B = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$U = \{A \in M_{2\times2}(\mathbb{R}) | trace(A) = 0\}$$

$$V = \mathcal{L} \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

(i) Prove that B is a basis of $M_{2\times2}(\mathbb{R})$.

(ii) Find the parametric and implicit equations of V with respect to the basis B.

(iii) Find the implicit equations of $U \cap V$ with respect to the basis B.

(Final exam, July 2017)

3.– In the vector space $\mathbb{R}^4$ we consider the subspaces

$$U = \mathcal{L}\{(1, 0, 1, 1), (2, 0, 1, 3), (1, 1, 1, 1)\}, \quad W = \{(x, y, z, t) \in \mathbb{R}^4 | x+y+z-t = 0, \; x+y-z+2t = 0\}.$$

(i) Find the parametric and implicit equations of $U \cap W$ with respect to the canonical basis.

(ii) Show that the vectors $B = \{(0, 1, 0, 0), (1, 1, 0, 0), (0, 0, 1, 1), (0, 0, 0, 1)\}$ form a basis of $\mathbb{R}^4$.

(iii) Find the parametric equations of U and the implicit equations of W with respect to the basis B.

(Final exam, January 2017)

4.– Let $\mathcal{P}_3(\mathbb{R})$ be the vector space of all polynomials of degree less than or equal to 3. Consider the following subsets:

$$U = \{p(x) \in \mathcal{P}_3(\mathbb{R}) | p(1) = 1\}, \qquad V = \{p(x) \in \mathcal{P}_3(\mathbb{R}) | p(1) = 0,\ p'(1) = 0\}$$

(i) Decide whether U and V are vector subspaces.

(ii) Show that the set $B = \{1, (x-1), (x-1)^2, (x-1)^3\}$ is a basis of $\mathcal{P}_3(\mathbb{R})$.

(iii) Find the implicit equations of V with respect to the basis B.

(Final exam, January 2018)

5.– Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all polynomials of degree less than or equal to 2. Consider the following subsets:

$$U = \{p(x) \in \mathcal{P}_2(\mathbb{R}) | p(1) = 0\}, \qquad V = \{p(x) \in \mathcal{P}_2(\mathbb{R}) | p(1) \cdot p(0) = 0\}, \qquad W = \mathcal{L}\{1 + x^2, 1 - x\}$$

(i) Determine which of the previous set are vector subspaces of $\mathcal{P}_2(\mathbb{R})$.

(ii) Find the implicit equations of U and W with respect to the canonical basis.

(iii) Find the parametric and implicit equations of $U \cap W$ with respect to the canonical basis.

(iv) Prove that $B = \{1, 1 + x, 1 + x + x^2\}$ is a basis of $\mathcal{P}_2(\mathbb{R})$.

(v) Find the implicit equations of $U \cap W$ with respect to the basis B.

(Final exam, January 2019)

6.– In the vector space $\mathbb{R}^3$ we consider the following bases:

- the canonical basis $C = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$.
- $B' = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\} = \{(0,1,1), (2,0,0), (1,0,1)\}$.
- $B'' = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\} = \{(1,-1,0), (0,0,-1), (1,1,1)\}$.

(a) If $(1,3,2)$ is a vector of $\mathbb{R}^3$, find its coordinates with respect to each of the previous bases.

(b) We denote by (y_1, y_2, y_3) the coordinates of a vector with respect to the basis B'. Consider the subspace defined by the equation:

$$y_1 + 2y_2 - y_3 = 0$$

Find the parametric and implicit equations of this subspace with respect to each of the given bases.

7.– In the vector space $\mathbb{R}^3$ we consider the set of vectors:

$$B = \{(1,1,1), (0,0,1), (1,0,1)\}$$

(i) Show that B is a basis of $\mathbb{R}^3$.

(ii) If $x' + y' + z' = 0$ is the implicit equation with respect to B of a subspace U, find its parametric equations with respect to the canonical basis.

(iii) If $x + y + z = 0$ is the implicit equation of another subspace V with respect to the canonical basis, find the parametric equations of $U \cap W$ with respect to the canonical basis B.

(Final exam, January 2022)

8.– In the vector space of matrices $M_{2\times 2}(\mathbb{R})$ we consider the following subsets:

$$U = \mathcal{L}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}. \qquad V = \{A \in M_{2\times 2}(\mathbb{R}) | trace(A) = 0,\ A = A^t\}.$$

(i) Prove that V is a vector subspace of $M_{2\times 2}(\mathbb{R})$.

(ii) Prove that U and V are complementary subspaces.

(iii) Find the projection of Id onto U along V.

(Final exam, January 2014)

9.– In the vector space $\mathbb{R}^3$ consider the vector subspaces:

$$U = \mathcal{L}\{(1,0,1),(2,0,1),(0,0,1)\}, \qquad V = \mathcal{L}\{(1,1,0),(1,2,3)\}$$

(i) Find the implicit equations of U and V with respect to the canonical basis.

(ii) Prove that the vectors of $B = \{(0,0,1),(1,2,1),(1,3,1)\}$ form a basis of $\mathbb{R}^3$.

(iii) Find the implicit and parametric equations of $U \cap V$ with respect to the basis B.

(iv) Are U and V complementary subspaces?

(v) Give the parametric equations of a subspace complementary to V with respect to the canonical basis.

(Exam, December 2014)

10.– In the vector space $\mathbb{R}^4$, given a real number a we consider the following subspaces:

$$U = \mathcal{L}\{(1,a,0,1),(2,2,a,0),(1,1,0,a)\}, \quad V = \{(x,y,z,t) \in \mathbb{R}^4 | x - y = 0, \quad x - z = 0\}.$$

(i) Find the dimensions of U, V, $U + V$ and $U \cap V$ in terms of a.

(ii) For $a = 1$ find the parametric and implicit equations of $U \cap V$.

(iii) For $a = 0$ prove that the subspaces U and V are complementary. Compute the projection of $(1,2,1,1)$ onto U along V.

(Final exam, july 2016)

11.– Let $\mathcal{P}_4(\mathbb{R})$ be the vector space of all polynomials of degree less than or equal to 4. Consider the subsets:

$$U = \{p(x) \in \mathcal{P}_4(\mathbb{R}) | p(-1) + p'(0) = 0\}$$
$$V = \{p(x) \in \mathcal{P}_4(\mathbb{R}) | p''(0) = 0\}$$

(i) Prove that they are vector subspaces.

(ii) Find the parametric and implicit equations of U, V, $U+V$, $U \cap V$ with respect to the canonical basis.

(iii) Are they complementary subspaces?

Notation $p'(x), p''(x)$ denote respectively the first and second derivatives of the polynomial $p(x)$.

12.– In $\mathbb{R}^4$ consider the vector subspaces:

$$U = \mathcal{L}\{(b, b, 1, 1), (1, 0, 1, 1), (1, 1, 1, 1)\}$$
$$V = \mathcal{L}\{(0, 0, 1, 1), (0, a, 1, 1), (0, 0, 0, 1)\}$$

(a) Find the dimension of $U \cap V$ in terms of a and b.

(c) For $a = 1$ and $b = 0$ write the implicit equations of $U \cap V$ with respect to the canonical basis.

(Final exam, June 2008)

13.– In the vector space $\mathcal{M}_{2\times 2}(\mathbb{R})$ we consider the subsets:

$$U = \{A \in \mathcal{M}_{2\times 2}(\mathbb{R}) | rank(A) < 2\}, \qquad V = \left\{A \in \mathcal{M}_{2\times 2}(\mathbb{R}) \,\middle|\, trace\left(A \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = 0\right\}$$
$$W = \mathcal{L}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$$

(i) Decide which of the previous subsets are subspaces of $\mathcal{M}_{2\times 2}(\mathbb{R})$.

(ii) Find the parametric and implicit equations of V with respect to the canonical basis.

(iii) Find the parametric and implicit equations of $V \cap W$ with respect to the canonical basis.

(iv) Prove that $B = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$ is a basis of $\mathcal{M}_{2\times 2}(\mathbb{R})$.

(v) Find the implicit equations of V with respect to the basis B.

(Final exam, July 2019)

14.– Let V be a real vector space and U_1, U_2, U_3 three vector subspaces of V. Determine and justify whether the following statements are true or false.

(a) $(U_1 + U_2) \cap (U_1 + U_3) \subset U_1 + (U_2 \cap U_3)$.

(b) $U_1 + (U_2 \cap U_3) \subset (U_1 + U_2) \cap (U_1 + U_3)$.

(First partial exam, January 2009)

15.– Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all polynomials of degree less than or equal to 2. Consider the subsets:

$$U = \{p(x) \in \mathcal{P}_2(\mathbb{R}) | p'(1) = p(0)\}, \quad V = \mathcal{L}\{1 + x^2\}$$

(i) Prove that U is a subspace of $\mathcal{P}_2(\mathbb{R})$.

(ii) Prove that U and V are complementary subspaces.

(iii) Calculate the polynomial that is the projection of $q(x) = (x - 1)^2$ onto V along U.

(Final exam, January 2022)

16.– Assume that the vector space V has a generating system made up of 2018 vectors. Determine and justify whether the following statements are true or false.

(i) V has a set of 2018 linearly independent vectors.

(ii) $dim(V) \geq 2018$.

(iii) $dim(V) \leq 2018$.

(iv) Any set of 2019 vectors of V is linearly independent.

(Final exam, July 2018)

I.– In the vector space $\mathcal{M}_{2\times 2}(\mathbb{R})$ we consider the subspaces:

$$U = \mathcal{L}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right\}$$
$$V = \{A \in \mathcal{M}_{2\times 2}(\mathbb{R}) | A = -A^t\}.$$

(i) Find the parametric and implicit equations of U and $U+V$ with respect to the canonical basis.

(ii) Are U and V complementary subspaces?

(iii) Prove that the following subset

$$B = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}\right\}$$

is a basis of $\mathcal{M}_{2\times 2}(\mathbb{R})$.

(iv) Find the parametric and implicit equations of U with respect to the basis B.

(Final exam, January 2016)

II.– Let V be a vector space. Given three vector spaces A, B, C, determine and justify whether the following statements are true or false.

○ $(A + B) \cap C \neq (A \cap C) + (B \cap C)$.

○ $(A + B) \cap C \subset (A \cap C) + (B \cap C)$.

○ $(A + B) \cap C \supset (A \cap C) + (B \cap C)$.

○ $(A + B) \cap C = (A \cap C) + (B \cap C)$.

(First partial exam, January 2007)

III.– Let $V = \{f : \mathbb{R} \longrightarrow \mathbb{R} | f \text{ continuous}\}$ be the vector space of all real continuous functions. Consider the subspace

$$W = \mathcal{L}\{sin^2(x/2), cos(x), cos(\frac{\pi}{2} - x), 2\}$$

(i) Prove that the vectors in $B = \{sin(x), cos(x), 1\}$ are linearly independent.

(ii) Prove that B is a basis of W.

(iii) Given $a \in \mathbb{R}$, prove that $sin(x + a) \in W$.

(iv) Obtain the coordinates of $sin(x + a)$ with respect to the basis B.

IV.– Let $\mathcal{M}_{n\times n}(\mathbb{R})$ be the vector space of real square matrices of order $n \times n$.

(a) Given a matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, prove that the set

$$S = \{B \in \mathcal{M}_{n\times n}(\mathbb{R}) : AB = \Omega\}$$

is a subspace of $\mathcal{M}_{n\times n}(\mathbb{R})$.

(b) If $n = 2$ and A has the form

$$\begin{pmatrix} \alpha & 1 \\ \beta & 1 \end{pmatrix}$$

where α, β are real numbers, determine in terms of α, β the dimension and a basis of S and the implicit equations of a complementary subspace of S.

V.– Let $\mathcal{S}_3$ be the vector space of real symmetric matrices of order 3. Decide which of the following subsets are subspaces. For each one of them, find a basis and its parametric and implicit equations with respect to the canonical basis and with respect to the basis

$$B' = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \right\}$$

(a) Regular matrices.

(b) Matrices with zero trace.

(c) Matrices whose first two rows are equal.

VI.– Let V be the vector space of real continous funcions defined on $[0, 1]$. Consider the following subsets:

$$U_1 = \left\{ f \in V \ : \ \int_0^1 f(x)\, dx = 0 \right\}$$

$$U_2 = \{ f \in V \ : \ f \ \text{ is } \ \text{constant} \}$$

(a) Prove that U_1 and U_2 are subspaces of V.

(b) Study if U_1 and U_2 are complementary subspaces of V.

(c) Find (if it exists) the projection of $h(x) = 1 + 2x$ onto U_1 along U_2.

(Final exam, September 2002)

VII.– In $\mathbb{R}^3$ and for each $a \in \mathbb{R}$, consider the vector subspaces

$$U = \mathcal{L}\{(a,1,0),(0,a,1),(1,0,-1)\}, \quad V = \mathcal{L}\{(a,0,-1),(a-1,0,2a)\}.$$

(a) Find the dimension of U, V, $U+V$ and $U \cap V$ in terms of the values of a.

(b) Find the values of a such that U and V are complementary subspaces.

(c) For the values of a for which it is possible, compute the associated matrix with respect to the canonical basis of the projection map onto U along V.

(d) For $a = 0$, is it possible to decompose $(1,1,1)$ as the sum of a vector in U and another in V?. If this decompositions exits, is it unique?

(First partial exam, January 2009)

VIII.– In $\mathcal{M}_{2\times 2}(\mathbb{R})$ and for a fixed $k \in \mathbb{R}$ we consider the following subspaces:

$$U = \mathcal{L}\left\{\begin{pmatrix} 1 & 1 \\ 0 & k \end{pmatrix}, \begin{pmatrix} k & 0 \\ k & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\right\}, \quad V = \mathcal{L}\left\{\begin{pmatrix} k & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & k-1 \\ 0 & 2 \end{pmatrix}\right\}$$

(i) Calculate in terms of k: $dim(U)$, $dim(V)$, $dim(U \cap V)$ and $dim(U+V)$.

(ii) For $k = 2$, find the parametric and implicit equations of $U+V$ with respect to the canonical basis.

IX.– Consider the subspaces U and W of $\mathbb{R}^3$ sucht that U is generated by the vectors $(1,0,1),(0,1,1),(1,1,2)$ and the implicit equation of W is $x - y + 2z = 0$. Obtain

(a) Basis of U, W, $U+W$ and $U \cap W$.

(b) The implicit equations of $U \cap W$.

(c) A basis of a subspace H complementary of $U \cap W$.

(d) The projection of $(2,3,5)$ onto $U \cap W$ along H.

(Final exam, June 2006)

X.– In the vector space $\mathbb{R}^3$, given two real values $a, b \in R$, we define the following subspaces:

$$U = \mathcal{L}\{(1,a,1),(b,1,a)\}, \qquad V = \mathcal{L}\{(0,1,1),(a-1,1,b)\}.$$

(a) Find the dimension of $U \cap V$ in terms of a and b.

(b) Find the values of a and b for which the subspaces are complementary.

(d) For $a = b = 0$ calculate the parametric and implicit equations of $U \cap V$.

(Final exam, September 2008)
