LINEAR ALGEBRA I — Problem Set 4

Equivalence of matrices. Systems of equations (Academic year 2022–2023)

1.– Find the reduced row echelon form of the following matrix:

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{pmatrix}$$

2.– Using elementary transformations, find the range, the canonical form B with respect to equivalence and nonsingular matrices P and Q satisfying $B = PAQ$, being A the matrix from the previous problem.

3.– Given the matrices $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,

(i) find (if it exists) an invertible matrix X such that $XA = B$.

(ii) find (if it exists) an invertible matrix Y such that $AY = B$.

(iii) Are A and B equivalent matrices?

(iv) Are the matrices AA^t and BB^t congruent?

(Final exam, July 2020)

4.– Given the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 1 & 1 & 8 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & a \\ 1 & 1 & 8 \\ 0 & 0 & b \end{pmatrix}.$$

(i) Find the values of a and b for which they are row equivalent.

(ii) Find the values of a and b for which they are equivalent.

(iii) For $a = -5$ and $b = 0$ find (if it exists) an invertible matrix X such that $XA = B$.

(iv) For $a = -5$ and $b = 0$ find (if it exists)an invertible matrix X such that $AX = B$.

(Final exam, January 2015)

5.– Given the matrices $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & 0 \\ 1 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & b & 1 \\ 2 & 5 & 1 \\ 1 & 1 & c \end{pmatrix}$.

(i) Find the values of a, b, c for which A and B are congruent.

(ii) Find the values of a for which there exists an invertible matrix P with $PAP^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. Give the corresponding matrix P for these values of a.

(Final exam, January 2020)

6.– Given the matrices

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -13 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

find a matrix $X \in M_{3\times3}(\mathbb{R})$ satisfying $XAX^t = B$.

(Final exam, January 2012)

7.– Consider the matrices $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & a \\ 2 & 2 & 2 \end{pmatrix}$. Find the values of a for which there exists an invertible matrix $X \in \mathcal{M}_{3\times3}(\mathbb{R})$ such that $XA = B$. Give the corresponding matrix X for these values of a.

(Final exam, January 2021)

8.– Let $X = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 2 & 2 & 2 & 0 \\ 2 & 0 & 2 & a \\ 1 & -1 & 1 & -2 \end{pmatrix}$.

(i) Find the values of a for which X and Y are row equivalent.

(ii) Find the values of a for which X and Y are column equivalent.

(Final exam, July 2017)

9.– Given the matrices $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & a \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

(i) Find the values of a and b for which the matrices A and B are congruent.

(ii) For $b = 1$ and $a = 0$ find an invertible matrix P such that $P^tAP = B$.

(Final exam, July 2019)

10.– Using elementary transformations, obtain the inverses of the following matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 2 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

11.– Discuss and (when possible) solve, in terms of the corresponding parameter(s), the following systems of equations:

$$\begin{cases} ax + y + z &= 1 \\ x + ay + z &= a \\ x + y + az &= a^2 \end{cases} \qquad \begin{cases} ax + ay &= b \\ bx + ay &= a \\ abx + aby &= 1 \end{cases}$$

12.– Find a system of linear equations whose solution is the following:

$$\begin{cases} x^1 &= 2\lambda - \mu \\ x^2 &= \lambda - 2\mu + \delta \\ x^3 &= -\lambda + \mu - 2\delta \\ x^4 &= \lambda + 2\delta \end{cases} \qquad (\lambda,\ \mu,\ \delta \in \mathbb{R})$$

13.– Given the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

(i) Find two invertible symmetric matrices 2×2 which are NEITHER congruent with A NOR congruent with each other.

(ii) Find the values of a for which the matrix $B = \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix}$ is congruent with A. For $a = -3$, give an invertible matrix P such that $PAP^t = B$.

(Final exam, January 2022)

14.– Give an example of three non-diagonal, invertible, symmetric matrices $A, B, C \in M_{3\times 3}(\mathbb{R})$, such that A and B are congruent but C is not congruent with A. Justify your answer.

(Final exam, July 2016)

15.– Let $A, B \in \mathcal{M}_{n\times m}(\mathbb{R})$ be two matrices. Justify the truth or falsehood of the following statements:

(i) If $rank(A) + rank(B) = 199$ then A and B are not row equivalent.

(ii) If A and B are row equivalent and they both have rank m then they are column equivalent.

(iii) If A and B are row equivalent then $A + B$ and $A - B$ are row equivalent.

(iv) $rank(AB) = rank(BA)$.

(Final exam, January 2020)

16.– Let $A, B \in M_{n\times m}(\mathbb{R})$ be two matrices. Justify the truth or falsehood of the following statements:

(i) If $rank(A) = n$ then $m \geq n$.

(ii) If $rank(A) = 1$ then all rows of A are null except for one of them.

(iii) If $n = m$ then $rank(A^2) = rank(A)$.

(iv) If B is invertible, then $rank(AB^t) = rank(A)$.

(Final exam, July 2021)

I.– Prove that the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 4 \end{pmatrix},$$

are congruent. Give an invertible matrix P such that $B = P^t A P$.

(Final exam, January 2018)

II.– Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 5 & -1 & 6 \\ -1 & 1 & -3 & -3 \end{pmatrix} \qquad M = \begin{pmatrix} m+1 & 0 & 1 & m \\ 1-m & m & 0 & 1 \\ m-1 & 1 & m & 0 \end{pmatrix}$$

Find the value of m for which A and M are equivalent.

(Extraordinary exam, December 2007)

III.– Given the matrices

$$A = \begin{pmatrix} 0 & 2 & 2 & 0 \\ 2 & 1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 0 & k \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Find the values of k for which there exists an invertible matrix $X \in M_{2\times 2}(\mathbb{R})$ such that $XA = B$, and obtain the corresponding matrix X in these cases.

(Final exam, January 2022)

IV.– Given the matrix

$$B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 0 & a & b \end{pmatrix}$$

(a) Find the values of a, b for which B is diagonalizable by congruence.

(b) Find the values of a, b for which B is congruent in $\mathbb{R}$ with

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

(c) Find the values of a, b for which B is congruent in $\mathbb{R}$ with the identity matrix.

(Final exam, January 2011)

V.– For each $k \in \mathbb{R}$ consider the matrices

$$A = \begin{pmatrix} k & 1 \\ k & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & k \\ k & 4 \end{pmatrix}.$$

Justify the truth or falsehood of the following statements.

i) If $k = 1$, then A and B are congruent.

ii) For $k = 2$, A and B are row equivalent.

iii) For $k = 2$, A and B are column equivalent.

(First partial exam, January 2009)

VI.– Consider the real matrices

$$A = \begin{pmatrix} -1 & 2 \\ 2 & 1 \\ 3 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & -3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Is it possible to find a matrix $X \in M_{2\times 2}(\mathbb{R})$ such that $AX = B$? What about a matrix $Y \in M_{3\times 3}(\mathbb{R})$ such that $YA = B$? Justify the answers.

VII.– Given $a, b \in \mathbb{R}$, find (using elementary transformations and in the cases for which it is possible) the inverse of the matrix:

$$\begin{pmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ -b & a & 0 & \cdots & 0 & 0 \\ 0 & -b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 \\ 0 & 0 & 0 & \cdots & -b & a \end{pmatrix}$$

VIII.– Obtain the canonical form of the following matrix with respect to congruence over the field $\mathbb{R}$ and over the field $\mathbb{C}$, as well as the corresponding transition matrices:

$$\begin{pmatrix} 6 & 2 & 0 \\ 2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$

IX.– Let $A, B \in M_{2\times 2}(\mathbb{R})$ be two matrices with the same determinant and the same trace. Is it possible for A and B to NOT be equivalent? What if they are also symmetric? Justify the answers.

(First partial exam, January 2010)

X.– Among the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(i) Which pairs of matrices are equivalent?

(ii) Which pairs of matrices are similar?

(iii) Which pairs of matrices are congruent? For each one of them, give the corresponding congruence transition matrix.

(Exam, July 2015)

XI.– Discuss and, when possible, solve the following system of equations in terms of the corresponding parameters:

$$\begin{cases} ax + 2z &= 2 \\ 5x + 2y &= 1 \\ x - 2y + bz &= 3 \end{cases}$$

XII.– Let $A, B \in \mathcal{M}_{2\times 2}(\mathbb{R})$. Justify the truth or falsehood of the following statements.

(i) If $det(A) \neq 0$ and moreover A and B are row equivalent then they are also column equivalent.

(ii) If $det(A) = det(B) = 0$ then A and B are equivalent.

(iii) $(A + B)(A - B) = A^2 - B^2$.

(iv) If A and B are congruent then $sign(trace(A)) = sign(trace(B))$.

(Exam, July 2019)

XIII.– Let $A, B \in \mathcal{M}_{2\times 2}(\mathbb{R})$, with $det(A) = 1$ and $det(B) = 2$. Justify the truth or falsehood of the following statements.

(i) A and B are congruent.

(ii) A and B can be congruent.

(iii) A and B can be similar.

(iv) If $A = Id$ and $trace(B) = 0$ then A and B are not congruent.

(Final exam, July 2017)

XIV.– Let $A, B \in \mathcal{M}_{2\times 2}(\mathbb{R})$ be symmetric matrices. Justify the truth or falsehood of the following statements.

(i) If A and B are congruent then they have the same number of positive entries in their diagonals, and the same number of negative entries as well.

(ii) If $sig(det(A)) = sig(det(B))$ then A and B are congruent.

(iii) $AB - BA$ is an antisymmetric matrix.

(iv) AB is a symmetric matrix.

(Final exam, January 2019)
