LINEAR ALGEBRA I — Problem Set 1

Sets and functions (Academic year 2024–2025)

1.– Given the following sets:

$$A = \{2, 3, 5, 7, 11\}$$
$$B = \{x \in Z | x \geq 4\}$$
$$C = \{x \in Z || x| < 5\}$$
$$D = \{x \in N | x \text{ is odd}\}$$

Calculate:

i) $(A \cap B) \cup C$.

ii) $(Z - D) \cap C$.

iii) $(C \cup A) \cap B$.

iv) $(A \cup (Z - B)) \cap (C \cup D)$.

2.– Given the sets X and Y we define the symmetric difference of X and Y to be the set:

$$X \Delta Y = (X \cup Y) - (X \cap Y)$$

(i) Let $A = \{a \in \mathbb{N} | 1 \leq a \leq 3\}$, $B = \{x \in \mathbb{N} | 1 < x^2 < 20\}$, $C = \{1, 2, 4\}$. Calculate $(A \cup B)\Delta C$, $A\Delta(B \cap C)$ and $C \Delta B$.

(ii) Discuss whether or not it is generally true that:

$$(X \cup Y)\Delta Z = (X \Delta Z) \cup (Y \Delta Z)$$

(Partial exam, October 2015)

3.– Let A, B, C be three sets. Determine whether the following statements are true or false. Explain your reasoning:

(i) $(A \cap B) \cup C = A \cap (B \cup C)$.

(ii) If $A \subset B$ then $A \cap C = B \cap C$.

(iii) If $A \cap C = C$ then $C \subset A$.

(iv) If $A \cup B = A \cap B$ then $A = B$.

(v) If $A \cap B = \emptyset$ then $(A \cup C) \cap (B \cap C) = \emptyset$.

(vi) If $A \cap B = B \cap C = \emptyset$ then $A \cap C = \emptyset$.

(Final exam, July 2015.)

4.– Let $f : X \to Y$ be a function, and let A, B be two subsets of X. Determine whether the following statements are true or false and for those that turn out not to be true, if they are true under the additional hypothesis that f is injective or surjective.

(a) $f(A \cup B) \subset f(A) \cup f(B)$

(b) $f(A) \cup f(B) \subset f(A \cup B)$

(c) $f(A \cap B) \subset f(A) \cap f(B)$

(d) $f(A) \cap f(B) \subset f(A \cap B)$

(e) $f(X \setminus A) \subset Y \setminus f(A)$

(f) $Y \setminus f(A) \subset f(X \setminus A)$

5.– Let $f : \mathbb{R} \to \mathbb{R}^+$ be defined as $f(x) = x^2$ and let $g : \mathbb{R}^+ \to \mathbb{R}$ be defined as $g(x) = +\sqrt{x}$. Is one function the inverse of the other?. Reason the answer.

6.– Given the following functions study if they are injective, surjective or bijective. Calculate also the inverse applications of those that turn out to be bijective..

(a) $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$, $x \mapsto \cos x + 1$

(a') $f_1 : \mathbb{R} \longrightarrow [0, 2]$, $x \mapsto \cos x + 1$

(a") $f_1 : [0, \pi] \longrightarrow [0, 2]$, $x \mapsto \cos x + 1$

(b) $f_2 : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $x \mapsto y$ si $y^2 = x$

(b') $f_2 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $x \mapsto y$ if $y^2 = x$

(c) $f_3 : \mathbb{R} \longrightarrow \mathbb{R}$, $x \mapsto \mathrm{tg} x$

(d) $f_4 : \mathbb{R} \longrightarrow \mathbb{R}$, $x \mapsto x^9$

(e) $f_5 : \mathbb{N} \longrightarrow \mathbb{N}$, $n \mapsto n!$

(f) $f_6 : \mathbb{R} \setminus \{3\} \longrightarrow \mathbb{R} \setminus \{2\}$, $x \mapsto \dfrac{2x - 4}{x - 3}$

(g) $f_7 : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \mapsto xy$

(h) $f_8 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x, y) \mapsto (x - y, x + y)$

(i) $f_9 : \mathbb{R} \setminus \{0\} \to \mathbb{R}^2$, $x \mapsto (x, 1/x)$

7.– Let A, B, C, D be sets, f a function from A to B, g a function from B to C and h a function from C to D. Prove that if $g \circ f$ and $h \circ g$ are bijective, then in fact f, g and h are bijective.

8.– Let $h : X \longrightarrow X$ be a function such that there exists a $n \in \mathbb{N}$ with $h^n = i_X$. Prove that h is bijective. (Note: $h^n = h \circ \overset{n}{\ldots} \circ h$; $\quad i_X$ is the identity map of X).

9.– Let $f : A \to B$ be a functions. Prove that:

(a) f is injective if and only if there is a map $g : B \to A$ such that $g \circ f = i_A$.

(b) f is surjective if and only if there is a map $h : B \to A$ such that $f \circ h = i_B$.

10.– Let X and Y be sets and $f : X \to Y$ a function. Prove that the following statements are equivalent:

(a) f is injective.

(b) For any subsets A, B of X such that $A \cap B = \emptyset$, $f(A) \cap f(B) = \emptyset$ holds.

(First partial, February 2003)
