

1.– Given the following sets

$$A = \{2, 3, 5, 7, 11\}$$

$$B = \{x \in \mathbb{Z} \mid x \geq 4\}$$

$$C = \{x \in \mathbb{Z} \mid |x| < 5\}$$

$$D = \{x \in \mathbb{N} \mid x \text{ is odd}\}$$

compute

i)  $(A \cap B) \cup C$ .

ii)  $(Z - D) \cap C$ .

iii)  $(C \cup A) \cap B$ .

iv)  $(A \cup (Z - B)) \cap (C \cup D)$ .

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2.– Given the sets  $X$  and  $Y$ , we define the symmetric difference of  $X$  and  $Y$  to be the set:

$$X \Delta Y = (X \cup Y) - (X \cap Y)$$

(i) Let  $A = \{a \in \mathbb{N} \mid 1 \leq a \leq 3\}$ ,  $B = \{x \in \mathbb{N} \mid 1 < x^2 < 20\}$ ,  $C = \{1, 2, 4\}$ . Compute  $(A \cup B) \Delta C$ ,  $A \Delta (B \cap C)$  and  $C \Delta B$ .

(ii) Discuss whether or not it is generally true that:

$$(X \cup Y) \Delta Z = (X \Delta Z) \cup (Y \Delta Z)$$

**(Partial exam, October 2015)**

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3.– Let  $A, B, C$  be three sets. Determine whether the following statements are true or false. Explain your reasoning:

(i)  $(A \cap B) \cup C = A \cap (B \cup C)$ .

(ii) If  $A \subset B$  then  $A \cap C = B \cap C$ .

(iii) If  $A \cap C = C$  then  $C \subset A$ .

(iv) If  $A \cup B = A \cap B$  then  $A = B$ .

(v) If  $A \cap B = \emptyset$  then  $(A \cup C) \cap (B \cap C) = \emptyset$ .

(vi) If  $A \cap B = B \cap C = \emptyset$  then  $A \cap C = \emptyset$ .

**(Final exam, July 2015.)**

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4.— Let  $f : X \rightarrow Y$  be a function, and let  $A, B$  be two subsets of  $X$ . Determine whether the following statements are true or false and for those that turn out not to be true, if they are true under the additional hypothesis that  $f$  is injective or surjective.

- (a)  $f(A \cup B) \subset f(A) \cup f(B)$
- (b)  $f(A) \cup f(B) \subset f(A \cup B)$
- (c)  $f(A \cap B) \subset f(A) \cap f(B)$
- (d)  $f(A) \cap f(B) \subset f(A \cap B)$
- (e)  $f(X \setminus A) \subset Y \setminus f(A)$
- (f)  $Y \setminus f(A) \subset f(X \setminus A)$

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5.— Let  $f : \mathbb{R} \rightarrow \mathbb{R}^+$  be defined as  $f(x) = x^2$  and let  $g : \mathbb{R}^+ \rightarrow \mathbb{R}$  be defined as  $g(x) = +\sqrt{x}$ . Is one function the inverse of the other?. Explain your reasoning.

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6.— Given the following functions study if they are injective, surjective or bijective. Calculate also the inverse applications of those that turn out to be bijective.

- (a)  $f_1 : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \cos x + 1$
- (a')  $f_1 : \mathbb{R} \rightarrow [0, 2], x \mapsto \cos x + 1$
- (a'')  $f_1 : [0, \pi] \rightarrow [0, 2], x \mapsto \cos x + 1$
- (b)  $f_2 : \mathbb{R}^+ \rightarrow \mathbb{R}, x \mapsto y$  si  $y^2 = x$
- (b')  $f_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^+, x \mapsto y$  if  $y^2 = x$
- (c)  $f_3 : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \operatorname{tg} x$
- (d)  $f_4 : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^9$
- (e)  $f_5 : \mathbb{N} \rightarrow \mathbb{N}, n \mapsto n!$
- (f)  $f_6 : \mathbb{R} \setminus \{3\} \rightarrow \mathbb{R} \setminus \{2\}, x \mapsto \frac{2x - 4}{x - 3}$
- (g)  $f_7 : \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto xy$
- (h)  $f_8 : \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x, y) \mapsto (x - y, x + y)$
- (i)  $f_9 : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}^2, x \mapsto (x, 1/x)$

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7.— Let  $A, B, C, D$  be sets,  $f$  a function from  $A$  to  $B$ ,  $g$  a function from  $B$  to  $C$  and  $h$  a function from  $C$  to  $D$ . Prove that if  $g \circ f$  and  $h \circ g$  are bijective, then in fact  $f, g$  and  $h$  are bijective.

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8.— Let  $h : X \rightarrow X$  be a function such that there exists a  $n \in \mathbb{N}$  with  $h^n = i_X$ . Prove that  $h$  is bijective. (Note:  $h^n = h \circ \dots \circ h$ ;  $i_X$  is the identity map of  $X$ ).

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9.— Let  $f : A \rightarrow B$  be a functions. Prove that:

- (a)  $f$  is injective if and only if there is a map  $g : B \rightarrow A$  such that  $g \circ f = i_A$ .
- (b)  $f$  is surjective if and only if there is a map  $h : B \rightarrow A$  such that  $f \circ h = i_B$ .

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10.— Let  $X$  and  $Y$  be sets and  $f : X \rightarrow Y$  a function. Prove that the following statements are equivalent:

- (a)  $f$  is injective.
- (b) If  $A, B$  are two subsets of  $X$  such that  $A \cap B = \emptyset$ , then  $f(A) \cap f(B) = \emptyset$ .

**(First partial exam, February 2003)**

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