LINEAR ALGEBRA I **Exercises Part III. Chapter 3**

Linear maps. (Curso 2022–2023)

1.– Given the function f:

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad f(x, y) = (x + 2y, x - y)$$

check that it is a linear map by verifying that the identity

$$f(a(x, y) + b(x', y')) = af(x, y) + bf(x', y')$$

holds for any a, b, x, y, x', y'.

2.– Given the function f:

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad f(x, y) = (x + 2y, 2)$$

check that it is NOT a linear map, by finding a vector (x, y) and a number a such that $f(a(x, y)) \neq af(x, y)$.

3.– Let $f : U \longrightarrow V$ be a linear map, $B_1 = \{\vec{u}_1, \vec{u}_2\}$ a basis of U and $B_2 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis of V. If $f(\vec{u}_1) = \vec{v}_1 + \vec{v}_2 - \vec{v}_3$ and $f(\vec{u}_2) = 2\vec{v}_1 + \vec{v}_3$, write the associated matrix $F_{B_2B_1}$ with respect to the bases B_1 of U and B_2 of V.

4.– Which is the image of the vector $\vec{w} = 5\vec{u}_1 - 3\vec{u}_2$ by the function from the previous problem?

5.– Under the conditions of Problem 3 we consider the new bases $B'_1 = \{\vec{u}'_1, \vec{u}'_2\}$ and $B'_2 = \{\vec{v}'_1, \vec{v}'_2, \vec{v}'_3\}$, with:

$$\vec{u}'_1 = \vec{u}_1 + \vec{u}_2, \quad \vec{u}'_2 = \vec{u}_1 + 2\vec{u}_2$$
$$\vec{v}'_1 = \vec{v}_2, \quad \vec{v}'_2 = \vec{v}_3, \quad \vec{v}'_3 = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$$

Find the matrix $F_{B'_2B'_1}$ with respect to the bases B'_1 of U and B'_2 of V.

6.– Given the linear map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined as $f(x, y, z) = (x + y + z, 2x - y, 3x + z)$ find the implicit equations of the kernel and a basis of the image.

7.– Given the linear maps $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $h = g \circ f$, defined as $f(x, y) = (x+y, 2x+y)$, $g(x, y) = (3x - y, x - y)$, find the matrices associated to f, g and h with respect to the canonical basis of $\mathbb{R}^2$.

Solutions.

2$^{(*)}$. $(x, y) = (0, 0)$ and $a = 0$. $f(0 \cdot (0, 0)) = (0, 2)$ but $0 \cdot f(0, 0) = (0, 0)$.

3. $F_{B_2B_1} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$.

4. $f(\vec{w}) = -\vec{v}_1 + 5\vec{v}_2 - 8\vec{v}_3$.

5. $F_{B'_2 B'_1} = \begin{pmatrix} -2 & -4 \\ -3 & -4 \\ 3 & 5 \end{pmatrix}.$

6$^{(*)}$. $ker(f) = \{(x,y,z) \in \mathbb{R}^3 | x+y+z=0, 2x-y=0\}$ and $Im(f) = \mathcal{L}\{(1,2,3),(1,-1,0)\}$.

7. $F_C = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad G_C = \begin{pmatrix} 3 & -1 \\ 1 & -1 \end{pmatrix}, \quad H_C = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}.$

$^{(*)}$ The solution is not unique, that is, there are several different correct answers.