LINEAR ALGEBRA I — Exercises Part II. Chapter 3 and 4

Equivalence of matrices. Systems of linear equations. (Academic year 2022–2023)

1.– Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$. If we successively apply on A the elementary row operations $H_{32}(1)$, $H_1(3)$, H_{13}, which matrix will we obtain?

2.– Write the elementary row matrices 3×3, $H_{32}(1)$, $H_1(3)$, H_{13}.

3.– For the matrix A from Exercise 1 compute the products $H_{32}(1)A$, $H_1(3)H_{32}(1)A$ and $H_{13}H_1(3)H_{32}(1)A$.

4.– In the set of 3×3 matrices, find the product $H_{13}H_1(3)H_{32}(1)$.

5.– If we successively apply to the 3×3 identity matrix, the elementary row operations $H_{32}(1)$, $H_1(3)$, H_{13}, which matrix will we obtain?

6.– Of the following matrices, which are reduced row echelon forms and which are reduced column echelon forms?

$$A_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \ A_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

7.– Find the reduced row echelon form R of the matrix A from Exercise 1.

8.– For the matrices A and R from Exercises 1 and 7, find the transition matrix P such that $PA = R$.

9.– Find the inverse of $C = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ by Gauss' method.

10.– Find a diagonal matrix D congruent with $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

11.– Apply on the identity matrix the same row operations as in Exercise 10 and check that the resulting matrix S satisfies $SBS^t = D$.

12.– Write the augmented matrix associated to the system:

$$\begin{cases} x + 2y + z = 4 \\ y + z = 1 \\ x + 3y + 2z = 5 \end{cases}$$

13.– Solve the above system by Gauss's method, giving the solution as a function of one parameter.

Solutions.

---

1. $A \xrightarrow{H_{32}(1)} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 3 & 4 & 4 \end{pmatrix} \xrightarrow{H_1(3)} \begin{pmatrix} 3 & 0 & 6 \\ 1 & 2 & 1 \\ 3 & 4 & 4 \end{pmatrix} \xrightarrow{H_{13}} \begin{pmatrix} 3 & 4 & 4 \\ 1 & 2 & 1 \\ 3 & 0 & 6 \end{pmatrix}.$

2. $H_{32}(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad H_1(3) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$

3. $H_{32}(1)A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 3 & 4 & 4 \end{pmatrix}, H_1(3)H_{32}(1)A = \begin{pmatrix} 3 & 0 & 6 \\ 1 & 2 & 1 \\ 3 & 4 & 4 \end{pmatrix}, H_{13}H_1(3)H_{32}(1) = \begin{pmatrix} 3 & 4 & 4 \\ 1 & 2 & 1 \\ 3 & 0 & 6 \end{pmatrix}.$

4. $H_{13}H_1(3)H_{32}(1) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \end{pmatrix}.$

5. $Id \xrightarrow{H_{32}(1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{H_1(3)} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{H_{13}} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \end{pmatrix}.$

6. A_2 and A_4 are reduced row echelon forms. A_3 and A_4 are reduced column echelon forms.

7. $R = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}.$

8$^{(*)}$**.** $P = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$

9. $C^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & -3 & 4 \\ 1 & 4 & -4 \end{pmatrix}.$

10$^{(*)}$**.** $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}.$

11$^{(*)}$**.** $Id \xrightarrow{H_{31}(-1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \xrightarrow{H_{23}(1)} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \xrightarrow{H_{32}(-1/2)} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1/2 & -1/2 & 1/2 \end{pmatrix} = S.$

12. $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 2 & 5 \end{array}\right).$

13. $x = 2 + \lambda, \qquad y = 1 - \lambda, \qquad z = \lambda, \qquad \lambda \in \mathbb{R}.$

$^{(*)}$ The solution is not unique, i. e. there may be different correct answers.