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Chapter 1

Preliminary considerations

1.1 Study of the movement of a fluid

The motion of a viscous fluid in three dimensions is described by the Navier-Stokes equations, a system of partial differential equations without analytical solution. When solving these equations numerically we may use different approaches. The most ambitious is the Direct Numerical Simulation that solves all fluid movements. In this case we need to use a mesh size at least as fine as the smallest eddies, which can lead, in a medium size problem, to a number of computing nodes the order of \(10^9\). Moreover, the frequency of the fastest events can be about \(10^4\) Hz which imposes a time step of not more than \(10^{-4}\) s to treat it properly.

A second approach, less expensive, is to simulate only the large eddies (Large Eddy Simulation, LES) modeling the effect of those of smaller size, which cannot be solved with a given mesh. The process can be described as “filtering equations”, after which the velocity field contains only the larger components. This process introduces a stress terms representing the interaction between the two scales of motion and have a dissipative effect. To calculate the effect of these stresses there are different models known in the literature as SGS (subgrid scale models). The LES is based on the work of Smagorinsky [24].

In engineering applications, there is usually no need to know all the details of a flow, but only some properties: the discharge through a channel, the velocity distribution in a section or a substance concentration in a certain volume. For these cases there is a third approach, far less expensive than the other two, that produces sufficiently accurate results: they are the Reynolds equations, which are obtained by averaging in time the Navier-Stokes equations (Reynolds Averaged Navier-Stokes, RANS). The time averaging of the
variables produce, similarly to the LES case, some terms called Reynolds stresses. The effect of these stresses can be modeled by estimating a turbulent viscosity, for which a turbulence model is needed.

1.2 Number of dimensions of the model

These models take into account the motion of fluid particles in the three spatial directions. This is a suitable approach to represent the three-dimensional reality, but the complexity of 3D models can be intractable even for relatively simple domains.

On the opposite end are the 1D models. They are accurate enough for certain phenomena, such as the movement of fluid in a pipe. The free surface, if it exists, is determined by the value of the variable depth \( h \). The equations are greatly simplified, leading to significant savings in computation time. The problem is that, in most cases, these models do not represent adequately the real problem, not permitting, for instance, taking into account the effect of a change of direction or an asymmetrical section.

However, there are a number of phenomena in which fluid motion occurs mainly in two dimensions, for example when the bottom slope is small and the movement of the particles is substantially parallel to it. This makes the 2D models an interesting option with considerable saving compared to 3D, and they allow an approximation to reality much greater than that achieved with 1D models.

1.3 Discretization techniques

With regard to the spatial discretization, most of the models used in Computational Fluid Dynamics (CFD) use one of the three following techniques: finite differences, finite elements or finite volumes.

The Finite Difference Method is the oldest of the three, although its popularity has declined, perhaps due to its lack of flexibility from the geometric point of view. It is usually applied to structured meshes. Its implementation is simple, so new numerical schemes can easily be developed (especially in 1D) that can be generalized to several dimensions and used in finite volume formulations.

The Finite Volume formulation is now widely used in computational fluid dynamics, being its use very common in the field of shallow water equations [3] and 3D models [33]. It is applied to both structured and unstructured meshes with different shapes of the volumes. Its flexibility and conceptual
simplicity explain the acceptance it has. It has been used in commercial programs [11]. In one dimension it is equivalent to the Finite Difference Method and, depending on the mesh used and the type of discretization, it can also be so with a higher number of dimensions.

The main advantage of the Finite Element Method stems from its rigorous mathematical foundation that allows a posteriori error estimation. It is conceptually more difficult than the Finite Volume Method and the physical meaning of the proceedings is less easily seen than in this, although its flexibility to adapt to any geometry is similar. It is used by different authors and applied to commercial programs [6].

Some works [19, 35] compare both methods, showing that the Finite Volume Method shares the theoretical basis of the Finite Element Method, since it is a particular case of the Weighted Residuals Formulation. However, the weighting used in the first (constant volumes in the case of first order approximation) allows to take advantage of some properties of conservation, and the resolution algorithms are posed in a very advantageous way.

1.4 Systems of hyperbolic equations

Hyperbolic equations systems have been studied by a number of authors over the last decades. In a first phase, the studies focused on homogeneous systems but, since the 80s, more interest has been put in problems with source term, which have more practical applications.

The applications were initially oriented to compressible fluids, achieving significant results in aerodynamics. The strong analogy between the equations of compressible and incompressible flows have permitted to apply similar techniques to the shallow water equations, e.g. Glaister [14] using finite differences, or Vázquez Cendón [31], with finite volumes. Donea and Huerta [12] apply the Finite Element Method, in permanent and non-permanent problems, both to compressible and incompressible fluids.

1.5 The Shallow Water Equations

The behavior of a viscous fluid is governed by the Navier Stokes equations. These equations were derived in 1821 by Claude Navier and, independently, by George Stokes in 1845. They form a hyperbolic system of nonlinear conservation laws and, due to their complexity, have no analytical solution. For this reason, the 2D system of Shallow Water (or Saint Venant) Equations has been obtained from them, by imposing several simplifying assumptions.
These equations describe the behavior of a fluid in shallow areas. Despite
the strong assumptions used in their obtaining, the results are very close to
reality, even in cases where some of these hypotheses are not fulfilled. Some
of the the many problems that can be solved are flow in channels and rivers,
tidal flows, sea currents or progression of shockwaves. The one-dimensional
version of these equations is commonly used in the study of flow in open
channels.

Despite its considerable simplicity compared to the Navier Stokes equa-
tions, even 1D Saint Venant equations have no analytical solution and must
be solved by approximate methods. The increase, in recent decades, of the
computer power has allowed an increasing use of the two-dimensional shallow
water equations.

Since the 70s of last century, the Finite Element Method has begun to
be applied to the shallow water equations: Zienkiewicz [34], and Peraire [22]
are among the authors who have worked on this line.

In parallel to this, the use of the Finite Volume method has grown: see,
for instance, the works of Vázquez Cendón [31] and Alcrudo and Garcia-

The calculation of the velocity field in a given domain permits the study
of many problems of practical interest, such as the sediment transport, the
evolution of the salt concentration in an estuary or the dispersal of pollutants.
Chapter 2

One dimensional approach

2.1 Introduction

The aim of this chapter is to show the main aspects of the method in one spatial dimension. First, several commonly used terms are defined and some basic concepts in numerical modeling are introduced or reminded. To describe some of the techniques, simple equations in 1D are used, such as the transport equation. In order to facilitate the application of the method to the particular case of the shallow water equations, the final chapter defines some terms commonly used in open channels hydraulics.

2.2 Conservative variables and conservation laws

Conservative variables. There is some freedom to choose the variables that describe the flow to study. One possible choice is to take the “primitive” or “physical” variables: the density $\rho$, the pressure $p$ and the three components of velocity, $u, v, w$. Another one is to use the so-called “conservative” variables, which result from applying the fundamental laws of conservation (of mass, momentum, energy). These variables are, for example: the three components of momentum per unit volume $\rho u, \rho v, \rho w$ and the total energy per unit volume. For systems of equations governing the free surface flow in one or two dimensions, such as the Shallow Water system, the conservative variables commonly used are the depth $h$ and its product by the velocity components: $hu$ in one dimension and $(hu, hv)$ in the two-dimensional case.

Conservation laws. They are systems of partial differential equations expressing conservation of $m$ quantities $u_1, \ldots, u_m$. If obtained from a control
volume fixed in space, which is crossed by the moving fluid, they are said
to be written in conservative form, this is the way that most resembles a
flow balance of mass and momentum [7, pg. 19]. If the control volume moves
with the fluid, so always contains the same particles, the non conservative
form is obtained [4, pg. 16]. A conservation law in conservative form is
written
\[ U_t + F_x = 0, \quad U = U(x, t), \quad F = F(U). \]  
(2.1)

\( U \) is called Variables Vector and \( F(U) \) Flux Functions Vector. When ex-
pressing the conservation laws in differential form, it is assumed that the
solutions satisfy the relevant requirements of regularity.

Example 1 (scalar): The Transport equation (linear advection equation).
\[ \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \quad u = u(x, t), \quad f(u) = au, \ a = \text{constant}. \]  
(2.2)

Example 2 (scalar): Burgers equation.
\[ \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \quad u = u(x, t), \quad f(u) = \frac{1}{2} u^2. \]  
(2.3)

Example 3 (system of conservation laws): the one-dimensional shallow
water equations.
\[ U = \left\{ \begin{array}{c} h \\
hu \end{array} \right\}, \quad F(U) = \left\{ \begin{array}{c} hu \\
hu^2 + \frac{1}{2} gh^2 \end{array} \right\}. \]  
(2.4)

Nonconservative Form. If, for instance, we replace \( f \) by its value in the
Burgers Equation we obtain the nonconservative form
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \quad u = u(x, t). \]  
(2.5)

If instead of an equation we consider a system of conservation laws, applying
the chain rule we obtain the following expression:
\[ \frac{\partial U}{\partial t} + \frac{dF}{dU} \frac{\partial U}{\partial x} = \frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0 \]  
(2.6)

where \( A \) is the Jacobian matrix
\[ A = \frac{dF}{dU} = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial u_1} & \cdots & \frac{\partial f_m}{\partial u_m} \end{pmatrix}. \]  
(2.7)
The nonconservative formulation (2.6) is equivalent to the conservative one (2.1), and has the same solution, provided that this solution is sufficiently regular. Otherwise the derivation which led to (2.6) is not valid. For example, if the solution is discontinuous—e.g., a shock wave—erroneous results are obtained.

**Integral Form.** The conservation laws can also be expressed in integral form. One reason for the use of this form is that the obtaining of the equations is based on physical conservation principles, generally expressed as integral relationships. On the other hand the integral formulation requires less derivability conditions on solutions, which allows to obtain discontinuous solutions. These discontinuous solutions do not verify the partial differential equation at every point because the derivatives are not defined at the discontinuities, and must meet a “jump condition” along them, which is obtained from the integral form (see 2.6.8, Rankine-Hugoniot condition). The solutions of the integral form are known as weak solutions.

### 2.3 The Riemann problem

We will analyze the Riemann problem for the importance it has on the Godunov method, from which the method that is described here derives.

The transport equation, already mentioned, has the form

\[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \quad u = u(x, t), \ a = \text{constante}. \]  

(2.8)

**The Cauchy problem.** applied to this equation consist in solving it with the initial condition

\[ u(x, 0) = u_0(x). \]  

(2.9)

As it can easily be seen by substituting in the equation, the solution is given by

\[ u(x, t) = u_0(x - at), \ \forall x \in \mathbb{R}, \ \forall t \geq 0, \]  

(2.10)

which can be interpreted saying that the function \( u \) moves in time, along the axis \( x \), speed \( a \) without deforming.

The points of the plane \( x, t \) in which the above said occurs are called characteristic curves. Their equation in this case is given by

\[ \frac{dx}{dt} = a \quad \text{(in general, } \frac{dx}{dt} = f'(u) \text{)} \]  

(2.11)

and in them the solution \( u \) of the equation remains constant. Indeed, if \( u \) is a solution of the equation, the total derivative satisfies

\[ \frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} = \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0. \]  

(2.12)
The Riemann problem is a particular case of the former characterized by the initial condition

\[ u(x, 0) = u_0(x) = \begin{cases} u_L & \text{si } x < 0, \\ u_R & \text{si } x > 0. \end{cases} \tag{2.13} \]

The initial discontinuity at \( x = 0 \) is propagated to a distance \( d = at \) at time \( t \). The characteristic curve \( x = at \) separates -in the plane \( x, t \)- the points in which the solution is \( u_L \) from those in which is \( u_R \) and it is represented in figure 2.2 in bold. The Riemann-problem that can be expressed briefly as \( \text{RP}(U_L, U_R) \)- has as solution

\[ u(x, t) = u_0(x - at) = \begin{cases} u_L & \text{si } x - at < 0, \\ u_R & \text{si } x - at > 0. \end{cases} \tag{2.14} \]

If, for example, \( a > 0 \), the wavefront will move to the right. The solution of the equation is \( u_L \) at all points that have already been reached by the wave, which are those to the left of the point \( x = at \) after time \( t \) (Figure 2.1). The characteristic curves thus represent the pairs \((x, t)\) corresponding to the advance of the wave. For example, at \( t = t^* \) and \( x = x^* \) the variable takes a certain value. To know the position of the point where the variable initially took the same value, we go down the curve passing through \((x^*, p^*)\) to find the horizontal axis (Figure 2.2). The result, unsurprisingly, is \( x = x^* - at^* \).
2.4 Centered and non-centered discretization

Before describing the finite volume method (section 2.6) and applying it to the shallow water equations (chapter 4), we apply the centered and non-centered discretization to the Transport Equation.

Equation (2.8) is considered again
\[
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \quad u = u(x,t), \quad a = \text{constant.} \tag{2.15}
\]

Different discretizations for this equation can be obtained from the Taylor series expansion. For example, if \(i\) is the spatial index and \(n\) the time index,
\[
u_{i}^{n+1} = u_{i}^{n} + \frac{\partial u}{\partial t} |_{i}^{n} \Delta t + \frac{1}{2} \frac{\partial^{2} u}{\partial t^{2}} |_{i}^{n} \Delta t^{2} + O(\Delta t^{3}) \tag{2.16}
\]
and the time derivative can be approximated as
\[
\frac{\partial u}{\partial t} |_{i}^{n} = \frac{u_{i}^{n+1} - u_{i}^{n}}{\Delta t} - \frac{1}{2} \frac{\partial^{2} u}{\partial t^{2}} |_{i}^{n} \Delta t + O(\Delta t^{2}), \tag{2.17}
\]
which is a first order forward discretization. Also
\[
u_{i+1}^{n} = u_{i}^{n} + \frac{\partial u}{\partial x} |_{i}^{n} \Delta x + \frac{1}{2} \frac{\partial^{2} u}{\partial x^{2}} |_{i}^{n} \Delta x^{2} + \frac{1}{6} \frac{\partial^{3} u}{\partial x^{3}} |_{i}^{n} \Delta x^{3} + O(\Delta x^{4}), \tag{2.18}
\]
\[
u_{i-1}^{n} = u_{i}^{n} - \frac{\partial u}{\partial x} |_{i}^{n} \Delta x + \frac{1}{2} \frac{\partial^{2} u}{\partial x^{2}} |_{i}^{n} \Delta x^{2} - \frac{1}{6} \frac{\partial^{3} u}{\partial x^{3}} |_{i}^{n} \Delta x^{3} + O(\Delta x^{4}), \tag{2.19}
\]
and subtracting these two equations, we obtain a space centered second order discretization
\[
\frac{\partial u}{\partial x} |_{i}^{n} = \frac{u_{i+1}^{n} - u_{i-1}^{n}}{2 \Delta x} - \frac{1}{6} \frac{\partial^{3} u}{\partial x^{3}} |_{i}^{n} \Delta x^{3} + O(\Delta x^{3}). \tag{2.20}
\]

Then equation (2.15) can be written in discretized form as
\[
\frac{u_{i}^{n+1} - u_{i}^{n}}{\Delta t} + a \frac{u_{i+1}^{n} - u_{i-1}^{n}}{2 \Delta x} = 0 \tag{2.21}
\]
from which the following numerical algorithm results
\[
u_{i}^{n+1} = u_{i}^{n} - \frac{1}{2} \frac{a \Delta t}{\Delta x} (u_{i+1}^{n} - u_{i-1}^{n}). \tag{2.22}
\]

This scheme, first order in time and second in space is called Euler explicit scheme and it can be shown that is unconditionally unstable [27, pg. 163].
CHAPTER 2. ONE DIMENSIONAL APPROACH

In order to remedy the lack of stability observed in the above scheme, we
discretize spatially in **non-centered form**, which produces two options
\[ \frac{\partial u}{\partial x} \bigg|_{i}^{n} = \frac{u_{i}^{n} - u_{i-1}^{n}}{\Delta x}, \quad (2.23) \]
\[ \frac{\partial u}{\partial x} \bigg|_{i}^{n} = \frac{u_{i+1}^{n} - u_{i}^{n}}{\Delta x}, \quad (2.24) \]
from which only one is successful, depending on the sign of the speed \( a \) of
the wave. If \( a > 0 \), the option (2.23) together with (2.17) results
\[ \frac{u_{i+1}^{n+1} - u_{i}^{n}}{\Delta t} + a \frac{u_{i}^{n} - u_{i-1}^{n}}{\Delta x} = 0 \quad (2.25) \]
from where
\[ u_{i}^{n+1} = u_{i}^{n} - c(u_{i}^{n} - u_{i-1}^{n}), \quad (2.26) \]
being
\[ c = \frac{a \Delta t}{\Delta x}, \quad (2.27) \]
This scheme proves to be stable [27, pg. 164] provided that
\[ 0 \leq c \leq 1. \quad (2.28) \]
The parameter \( c \) is called the **Courant number** or the CFL (Courant-
Friedrichs-Lewy) number. It can be considered as the ratio of two lengths:
the one traveled by the wave in time \( \Delta t \) and the mesh size \( \Delta x \). As \( a \) is a
datum and \( \Delta x \) is usually determined by the desired degree of accuracy, one
can only vary \( \Delta t \) to satisfy the stability condition.

The scheme (2.26) is known as the **first order upwind scheme** and also
the CIR scheme, (Courant, Isaacson and Rees). The name upwind refers to
the fact that in the spatial discretization we use grid points from the side
where information comes. The CIR has the disadvantage, common to all first
order methods of being very diffusive: it tends to smooth discontinuities in
the solution and cut extreme values.

If, for \( a > 0 \), we introduce (2.17) and (2.24) in the transport equation
(2.15), the resulting downwind scheme
\[ u_{i}^{n+1} = u_{i}^{n} - c(u_{i+1}^{n} - u_{i}^{n}), \quad (2.29) \]
is unconditionally unstable. That is, to obtain a useful non-centered scheme
the sign of \( a \) in the spatial discretization must be taken into account.
Another first order scheme is the Lax-Friedrichs, characterized for replacing the term $u^n_i$ in (2.22) by

$$\frac{1}{2}(u^n_{i-1} + u^n_{i+1}),$$  \hspace{1cm} (2.30)

i.e. the average of the values in the two neighboring nodes. The resulting scheme

$$u^{n+1}_i = \frac{1}{2}(1 + c)u^n_i - \frac{1}{2}(1 - c)u^n_{i+1}$$  \hspace{1cm} (2.31)

is stable under the condition (2.28) [27, pg. 168].

### 2.5 Numerical diffusion or viscosity

We will try to clarify below the previous section assertion that the first order schemes are diffusive. Let us consider Equation (2.8) without time derivative

$$\frac{a}{\partial x} = 0, \hspace{0.5cm} a > 0.$$  \hspace{1cm} (2.32)

If the non-centered discretization (2.23) is applied, we get

$$\frac{a}{\partial x} \bigg|_i = a \frac{u_i - u_{i-1}}{\Delta x} + O(\Delta x).$$  \hspace{1cm} (2.33)

Moreover, from the Taylor series expansion (2.19),

$$u_{i-1} = u_i - \frac{\partial u}{\partial x} \bigg|_i \Delta x + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \bigg|_i \Delta x^2 + O(\Delta x^3),$$  \hspace{1cm} (2.34)

whence, multiplying by $a$ and rearranging, we obtain

$$a \frac{\partial u}{\partial x} \bigg|_i - \frac{1}{2} a \Delta x \frac{\partial^2 u}{\partial x^2} \bigg|_i = a \frac{u_i - u_{i-1}}{\Delta x} + O(\Delta x^2).$$  \hspace{1cm} (2.35)

That is, the expression

$$a \frac{u_i - u_{i-1}}{\Delta x},$$  \hspace{1cm} (2.36)

which represents a first order discretization of (2.32), is simultaneously a second order discretization (thus more accurate) of

$$a \frac{\partial u}{\partial x} - \frac{a \Delta x}{2} \frac{\partial^2 u}{\partial x^2} = 0,$$  \hspace{1cm} (2.37)

containing a diffusive term with a coefficient $a \Delta x/2$. 
So when discretizing upwind Equation (2.32) a so-called numerical diffusion is being introduced. The coefficient that quantifies this diffusion (also called numerical viscosity) depends on the mesh size, so if $\Delta x$ is sufficiently small, thereby increasing the computation time, the diffusive effect tends to disappear. If, however, the diffusive effect is high, the extreme values of the solution tend to cut and discontinuities to spread.

Another solution to reduce the diffusive effect is to use higher order schemes. These schemes take into consideration the values in a larger number of nodes, so the programming is more complicated. Lowering the numerical viscosity also reduces stability.

2.6 Conservative schemes.

The finite volume method in one spatial dimension is based on dividing the spatial domain into intervals (called finite volumes or cells) making in each of them an approximation of the integral of the conservative variables. At each time step these values are updated using approximations of the flux at the ends of the intervals, as it will be discussed below, using the scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \quad u = u(x,t), \quad f = f(u),$$  \hspace{1cm} (2.38)

that represents the transport equation if $f(u) = au$, being $a$ a constant.

2.6.1 Integral Form

A way to discretize (2.38), considering weak solutions, is to divide the spatial domain into finite volumes and integrate the equation in each cell, transforming it into an integral form. For simplicity, we will use intervals (finite volumes) with equal length $\Delta x$ and take a constant time step $\Delta t$. Thus the spatial and temporal domains will be

$$I_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right] = \left[x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2}\right],$$

$$I_n = [t_n, t_{n+1}] = [n\Delta t, (n+1)\Delta t]$$  \hspace{1cm} (2.39)

and the integral in the cell, of Equation (2.38)

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left[\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x}\right] dx = 0,$$  \hspace{1cm} (2.40)
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becomes
\[ \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial u}{\partial t} dx + f(u(x_{i+1/2}, t)) - f(u(x_{i-1/2}, t)) = 0. \]  \hspace{1cm} (2.41)

Since the interval ends \( x_{i \pm 1/2} \) do not depend on time, we can write
\[ \frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t) dx + f(u(x_{i+1/2}, t)) - f(u(x_{i-1/2}, t)) = 0. \]  \hspace{1cm} (2.42)

We define \( u^n_i \) as the spatial average of the function \( u(x, t) \) in the interval \( I_i \), at time \( t_n = n \Delta t \), i.e.
\[ u^n_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t_n) dx, \]  \hspace{1cm} (2.43)

Integrating (2.42) between \( t_n \) and \( t_{n+1} \), the time derivative disappears from the first term, resulting
\[ \int_{x_{i-1/2}}^{x_{i+1/2}} [u(x, t_{n+1}) - u(x, t_n)] dx \
+ \int_{t_n}^{t_{n+1}} \left[ f(u(x_{i+1/2}, t)) - f(u(x_{i-1/2}, t)) \right] dt = 0 \]  \hspace{1cm} (2.44)

and we see that the value of \( u \) in \( I_i \) only changes along time \( \Delta t \) due to the value of the flux \( f \) at the ends of \( I_i \). Then, using (2.43),
\[ (u^n_{i+1} - u^n_i) \Delta x + \int_{t_n}^{t_{n+1}} \left[ f(u(x_{i+1/2}, t)) - f(u(x_{i-1/2}, t)) \right] dt = 0. \]  \hspace{1cm} (2.45)

2.6.2 Numerical fluxes

In the above expression, the values of the integral of \( f \) at points \( x_{i \pm 1/2} \) will not be generally known, so we replace them with
\[ f^n_{i \pm 1/2} \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(u(x_{i \pm 1/2}, t)) dt, \]  \hspace{1cm} (2.46)

so we get
\[ u^n_{i+1} = u^n_i - \frac{\Delta t}{\Delta x} \left( f^n_{i+1/2} - f^n_{i-1/2} \right). \]  \hspace{1cm} (2.47)

The explicit algorithm (2.47) allows us to obtain the approximation of \( u \) in each cell, at time \( t_{n+1} \), from its value in the previous time and the numerical fluxes \( f^n_{i \pm 1/2} \) at the ends of the cell.
These numerical fluxes represent approximations of the time average of the physical flux at the edges of the cell and, depending on the way they are calculated, we get different schemes. To calculate them, the variables in cells adjacent to $I_i$ are used

$$f_{i+\frac{1}{2}}^n = \phi(u_{i-m}^n, u_{i-m+1}^n, \ldots, u_{i+1}^n), \quad \text{ (2.48)}$$

where $m$ and $l$ are two non-negative integers and $\phi$ is a certain function.

In hyperbolic problems information propagates at a finite speed, so it seems reasonable to assume that we can obtain $f_{i-\frac{1}{2}}^n$ from $u_{i-1}^n$ and $u_i^n$ (the average values of the variable on both sides of the boundary $x_{i-\frac{1}{2}}$), while $f_{i+\frac{1}{2}}^n$ is obtained from $u_i^n$ and $u_{i+1}^n$. Then the general expression (2.48) takes the form

$$f_{i-\frac{1}{2}}^n = \phi(u_{i-1}^n, u_i^n); \quad f_{i+\frac{1}{2}}^n = \phi(u_i^n, u_{i+1}^n). \quad \text{ (2.49)}$$

### 2.6.3 Convergence

The algorithm (2.47) allows us to obtain variable values forward in time. To provide a good approximation of the law of conservation, the algorithm must be convergent, which means that the numerical solution converges to the solution of the differential equation when $\triangle x, \triangle t \to 0$.

Convergence is ensured with two requirements: consistency and stability. Indeed, Lax Theorem states that a consistent and stable scheme is convergent. We will briefly discuss both.

### 2.6.4 Consistency condition

We say that a scheme is **consistent** if it represents faithfully the differential equation when $\triangle t \to 0$, $\triangle x \to 0$. As we are getting the numerical flux from the values of $u$ in neighboring cells, if $u$ has the same value in all of them, the result must be the same in each one. Therefore, a consistency condition required to function $\phi$ is:

$$\phi(v, v, \ldots, v) = f(v); \quad \text{ (2.50)}$$

Usually, continuity for the variable $u$ is also required, i.e. $\phi(u_{i-1}^n, u_i^n) \to f(v)$, when $u_{i-1}^n, u_i^n \to v$ [20, pg. 68].
2.6.5 Stability condition

A method must be stable in the sense that a small error introduced at any time step is not amplified indefinitely but remains bounded along the process. In paragraph 2.4 it was said that the Euler explicit scheme was unconditionally unstable, while the CIR scheme was stable when \(0 \leq c \leq 1\). These statements are based on the stability criterion of Von Neumann, which is based on Fourier analysis and is very useful in the study of linear systems.

2.6.6 Conservative scheme

A conservative scheme for the scalar conservation law (2.38) is a numerical method of the form (2.47) that fulfills the condition (2.48).

We see that by applying a conservative scheme to a set of contiguous cells \(N, N+1 \ldots M\), the result verifies the same property (2.44) of the exact solution (the value of \(u\) in \(I_i\) only changes in time \(\Delta t\) due to the value of the flow \(f\) at the ends of \(I_i\)). Indeed, adding the values of \(u^{n+1}_i\) obtained from (2.47), for any set of consecutive cells, multiplying by \(\Delta x\) and rearranging, we get

\[
\left( \sum_{i=N}^{M} u^{n+1}_i - \sum_{i=N}^{M} u^n_i \right) \Delta x + \left( f^n_{M+\frac{1}{2}} - f^n_{N-\frac{1}{2}} \right) \Delta t = 0,
\]

(2.51)

since fluxes at cell boundaries cancel each other, except for flows at the ends \(x = x_{N-\frac{1}{2}}\) and \(x = x_{M+\frac{1}{2}}\).

The interest of conservative schemes is that, as the Theorem of Lax-Wendroff [15, pg. 168] says, if a consistent conservative scheme converges, the result is a weak solution of the equation. In contrast, non-conservative schemes may not converge to the correct solution, if a shock wave is present [27, pg. 170]. Two examples of conservative schemes are the Godunov and Lax-Friedrichs schemes.

We may say that the Lax-Wendroff theorem “continues” Lax’s Theorem. That is, a scheme consistent, stable and conservative converges at a weak solution of the equation.

The algorithm (2.47) can also be seen as a finite difference approximation of the conservation law (2.38), as this law can be discretized as

\[
\frac{u^{n+1}_i - u^n_i}{\Delta t} + \frac{f^n_{i+\frac{1}{2}} - f^n_{i-\frac{1}{2}}}{\Delta x} = 0,
\]

(2.52)

where \(f^n_{i+\frac{1}{2}}, f^n_{i-\frac{1}{2}}\) are approximations of the value of \(f\) at the endpoints.
2.6.7 Godunov Method for a scalar equation

Godunov conducted the first conservative extension of the CIR scheme to nonlinear systems of conservation laws. The Godunov first order upwind method is a conservative scheme in the form (2.47), where the numerical fluxes at the boundaries of the cells, $f_{i+\frac{1}{2}}$, are calculated using solutions of local Riemann problems. It means that a Riemann Problem is solved in every time step at every boundary between two cells, taking as initial values at each side of the boundary, the average values of the variable in the previous time step, as discussed below.

It is assumed that in each time step $t_n$, variable $u$ is piecewise constant, taking on each cell $I_i$ the value given by (2.43). There are, then, a pair of steady states at each boundary of $I_i$: $(u_{i-1}, u_i)$ on the left and $(u_i, u_{i+1})$ on the right, both of which can be considered as a local Riemann Problem, originating at $x = 0, t = 0$. Thus, in the left side, $x = x_{i-\frac{1}{2}}$, we have

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0,$$

$$u(x, 0) = u_0(x) = \begin{cases} u^n_{i-1}, & x < 0, \\ u^n_i, & x > 0, \end{cases} \quad (2.53)$$

and, on the right side, $x = x_{i+\frac{1}{2}}$,

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0,$$

$$u(x, 0) = u_0(x) = \begin{cases} u^n_i, & x < 0, \\ u^n_{i+1}, & x > 0. \end{cases} \quad (2.54)$$

Let $\tilde{u}(x, t)$ be the combined solution of $RP(u^n_{i-1}, u^n_i)$ and $RP(u^n_i, u^n_{i+1})$ in $I_i$. Since $\tilde{u}(x, t)$ is the exact solution of the conservation law (2.38), we introduce it in the integral form (2.44), with spatial and temporal domains, respectively

$$I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], \quad I_n = [0, \Delta t], \quad (2.55)$$

obtaining

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tilde{u}(x, \Delta t) dx = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tilde{u}(x, 0) dx$$

$$- \int_0^{\Delta t} f(\tilde{u}(x_{i+\frac{1}{2}}, t)) dt + \int_0^{\Delta t} f(\tilde{u}(x_{i-\frac{1}{2}}, t)) dt = 0. \quad (2.56)$$
Now we define, as in (2.43)

\[ u^n_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \hat{u}(x, 0) dx, \tag{2.57} \]

\[ u^{n+1}_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \hat{u}(x, \Delta t) dx, \]

and it results the conservative scheme (2.47)

\[ u^{n+1}_i = u^n_i - \frac{\Delta t}{\Delta x} \left( f_{i+1/2}^n - f_{i-1/2}^n \right), \tag{2.58} \]

being

\[ f_{i\pm 1/2} = \frac{1}{\Delta t} \int_0^{\Delta t} f(\hat{u}(x_{i\pm 1/2}, t)) dt. \tag{2.59} \]

The integrand in (2.59) depends on the exact solution to the Riemann Problem, at each end of the cell, along the time axis (in local coordinates, then at \( x = 0 \)). This is represented as

\[ \hat{u}(x_{i\pm 1/2}, t) = u_{i\pm 1/2}^n(0), \tag{2.60} \]

whereby

\[ f_{i\pm 1/2} = f(u_{i\pm 1/2}^n(0)). \tag{2.61} \]

If, for instance, the flux function is \( f(u) = au, \ a > 0 \), it results

\[ f_{i-1/2} = au_{i-1}^n, \ f_{i+1/2} = au_i^n \tag{2.62} \]

(if, instead, we take \( a < 0 \) the result is \( au_i^n \) on the left boundary and \( au_{i+1}^n \) on the right). Replacing in (2.58), we arrive at

\[ u^{n+1}_i = u^n_i + \frac{\Delta t}{\Delta x} (au_{i-1}^n - au_i^n) = u^n_i - \frac{a\Delta t}{\Delta x} (u_i^n - u_{i-1}^n), \tag{2.63} \]

i.e. the CIR scheme (2.26).

Thus, Godunov Method considers the problem to be solved as a succession of states, constant in each finite volume. At each time step a Riemann problem at the boundary of each cell is solved, taking the exact solutions of each local problem as the fluxes in these boundaries. These exact solutions must be calculated according to the equation in question, if it is not linear; in [27, pg.176], the exact solutions of Riemann problem in the
case of the quasi-linear Burgers equation may be seen. Finally, the spatial averaging of the dependent variables in each cell is performed.

To simplify the process, different authors [16, 23, 30] have used schemes called approximate Riemann solvers, which they have applied to compressible fluids. These schemes have been extended later [2, 14, 29, 31] to free surface flows with very good results (in [20] different approximate Riemann solvers are described). While it is true that these schemes will replace the exact solution of the Riemann problem by an approximate one, the information provided by the exact solution is partially lost in any case, due to spatial averaging in each cell, which makes less significant the error in the approximation [8].

In the conservative scheme of Lax-Friedrichs the fluxes at the ends are calculated as

$$f_{i+\frac{1}{2}} = \frac{1+c}{2c} f(u^n_i) + \frac{c-1}{2c} f(u^n_{i+1}),$$

$$f_{i-\frac{1}{2}} = \frac{1+c}{2c} f(u^n_{i-1}) + \frac{c-1}{2c} f(u^n_i),$$

where $c$ takes the value given by (2.27). If $f(u) = au$ and we replace (2.64) in equation (2.47), the finite differences scheme (2.31) of Lax-Friedrichs is obtained.

### 2.6.8 Rankine-Hugoniot jump condition

In the preceding description of Godunov’s Method, the integral form is discretized, looking for weak solutions to the differential equation (2.38), when the initial condition is discontinuous (Riemann problem).

Of course, any function $u$ that is a “classical” solution (hence differentiable) of the equation will be a weak solution. And a weak solution is a “classical” solution in the intervals in which it is differentiable.

In the event that there is a discontinuity in a weak solution $u(x, t)$, the function will take values $u_L$ and $u_R$ at both sides of the discontinuity. Then the following relationship, known as the Rankine-Hugoniot condition, is verified,

$$(u_R - u_L)S = f(u_R) - f(u_L)$$

being $S$ the speed at which the jump is transmitted.

The Rankine-Hugoniot condition is shown in [27, pg. 70], using the Leibnitz formula and in [20, pg. 212] based on geometrical considerations. In the following examples we obtain this velocity $S$ in two cases.

Example 1. The transport equation, where $f(u) = au$, a constant.

$$S = \frac{f(u_R) - f(u_L)}{u_R - u_L} = \frac{au_R - au_L}{u_R - u_L} = a.$$
As we already knew (2.3), the wave moves at a constant speed \( a \).

Example 2. Burgers equation. Now \( f(u) = \frac{1}{2}u^2 \).

\[
S = \frac{1}{2}u_R^2 - \frac{1}{2}u_L^2 \quad \text{and} \quad \frac{u_R - u_L}{2}.
\] (2.67)

In this case, the speed of advance of the wave depends on the values of \( u \) at both sides of the discontinuity.

### 2.7 Hyperbolic linear systems

In paragraphs 2.4 to 2.6 we have referred to the case of a partial differential equation. Conservation laws are usually given in the form of a system of nonlinear equations. We will describe some techniques, for the case of linear systems, which can be extended to nonlinear systems.

Let a linear system of partial differential equations

\[
U_t + AU_x = 0, \quad U = \{u_j\}, \quad j = 1, 2 \ldots m,
\] (2.68)

where \( U \) is the variables vector and \( A_{m \times m} \) a constant matrix. The system is called hyperbolic if \( A \) is diagonalizable, i.e. if it has \( m \) real eigenvalues \( \lambda_i \) and \( m \) eigenvectors linearly independent \( k_i \). It is strictly hyperbolic if all the eigenvalues are different.

Calling \( \Lambda \) the diagonal matrix of eigenvalues and \( X \) the matrix whose columns are the eigenvectors \( k_i \), it holds

\[
A = \Lambda X X^{-1}.
\] (2.69)

The existence of \( X^{-1} \) allows us to define a new vector of variables

\[
V = \{v_j\}_{j=1,2 \ldots m} = X^{-1}U.
\] (2.70)

Using the relationships (2.69) and \( U = XV \) it results, from (2.68),

\[
XV_t + X\Lambda X^{-1}XV_x = XV_t + X\Lambda V_x = X(V_t + \Lambda V_x) = 0,
\] (2.71)

from where

\[
V_t + \Lambda V_x = 0.
\] (2.72)

Thus the system in canonical or characteristic form has been obtained. Each of the resulting \( m \) uncoupled equations have only one variable involved

\[
\frac{\partial v_j}{\partial t} + \lambda_j \frac{\partial v_j}{\partial x} = 0 \quad j = 1, 2 \ldots m,
\] (2.73)
and take the form of the 1D transport equation. Thus, the system can be seen as a combination of \(m\) waves traveling at their characteristic velocities given by the \(m\) eigenvalues \(\lambda_j\). These eigenvalues (or characteristic values) define the characteristic curves \(x(t) = x_0 + \lambda_j t\), (see 2.3, along which the information corresponding to each one of the characteristic variables \(\{v_j\}\) propagates.

With the above said we have made a basis change to the set of eigenvectors \(k^i\). The characteristic variables can be interpreted then as the components of the variables vector \(U\) in the reference system formed by the eigenvectors (being \(X^{-1}\) the basis change matrix) or, what is the same, as the projections of the vector \(U\) on the eigenvectors.

After solving the system (2.73), the values \(U\) can be obtained from the relationship (2.70). For example, let us consider the system

\[
U_t + F_x = 0, \quad U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad F = \begin{bmatrix} 2u_1 \\ u_1 + u_2 \end{bmatrix}
\]

which, by applying the rule of the chain, changes to

\[
U_t + AU_x = 0, \quad U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.
\]

From matrix \(A\) we obtain

\[
\lambda_1 = 2, \quad \lambda_2 = 1, \quad X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad X^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.
\]

Then, using the base change (2.70), the characteristic variables are found to be

\[
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 - u_1 \end{bmatrix},
\]

and, using these variables, we obtain the decoupled system

\[
\frac{\partial v_1}{\partial t} + 2 \frac{\partial v_1}{\partial x} = 0 \quad (2.78)
\]
\[
\frac{\partial v_2}{\partial t} + 1 \frac{\partial v_2}{\partial x} = 0. \quad (2.79)
\]

Then, if the initial condition of the problem is

\[
v_1(x, 0) = v_1^0(x); \quad v_2(x, 0) = v_2^0(x),
\]

the system solution will be, as in (2.10)

\[
v_1(x, t) = v_1^0(x - 2t), \quad v_2(x, t) = v_2^0(x - t), \quad \forall x \in R, \quad \forall t \geq 0
\]

and we can undo the variables change by using the matrix \(X\)

\[
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_1 + v_2 \end{bmatrix}. \quad (2.82)
\]
2.8 Non-centered schemes for linear systems

As we have just seen, the linear hyperbolic system (2.68)

\[ \mathbf{U}_t + \mathbf{A} \mathbf{U}_x = 0 \]

can decouple, becoming \( m \) equations (2.73), each one involving a single variable. But the non-centered conservative scheme studied in the scalar case (2.38) can only be applied to this system if all eigenvalues have the same sign. Then, in a general case, with eigenvalues of both signs, we will discretize upwind for the positive ones and downwind for the negatives. The practical way to accomplish this is to decompose the matrix \( \mathbf{A} \) into two, one with positive or zero eigenvalues and the other with negative or zero eigenvalues.

**Scheme CIR.** Calling

\[
\lambda_j^+ = \begin{cases} 
\lambda_j & \text{si } \lambda_j \geq 0, \\
0 & \text{si } \lambda_j < 0,
\end{cases} \quad (2.83)
\]

\[
\lambda_j^- = \begin{cases} 
\lambda_j & \text{si } \lambda_j \leq 0, \\
0 & \text{si } \lambda_j > 0,
\end{cases} \quad (2.84)
\]

the CIR scheme (2.26) applied to a decoupled hyperbolic linear system (in canonical form) is

\[
\{v_j\}_{i}^{n+1} = \{v_j\}_{i}^{n} - \frac{\Delta t}{\Delta x} \lambda_j^+ (\{v_j\}_{i}^{n} - \{v_j\}_{i-1}^{n}) - \frac{\Delta t}{\Delta x} \lambda_j^- (\{v_j\}_{i+1}^{n} - \{v_j\}_{i}^{n}),
\]

\[ j = 1, 2 \ldots m. \] \( (2.85) \)

Let \( \mathbf{\Lambda} \) be the diagonal matrix of eigenvalues \( \lambda_j \), \( \mathbf{\Lambda}^+ \) of the \( \lambda_j^+ \), \( \mathbf{\Lambda}^- \) of the \( \lambda_j^- \) and \( |\mathbf{\Lambda}| \) of the absolute values \( |\lambda_j| \). It holds

\[
\mathbf{\Lambda} = \mathbf{\Lambda^+ + \Lambda^-}, \quad (2.86)
\]

\[
|\mathbf{\Lambda}| = \mathbf{\Lambda^+ - \Lambda^-}. \quad (2.87)
\]

If we define

\[
\mathbf{\Lambda^+} = \mathbf{X}\mathbf{\Lambda^+ X^{-1}}, \quad (2.88)
\]

\[
\mathbf{\Lambda^-} = \mathbf{X}\mathbf{\Lambda^- X^{-1}}, \quad (2.89)
\]

\[
|\mathbf{\Lambda}| = \mathbf{X}|\mathbf{\Lambda}|\mathbf{X^{-1}}, \quad (2.90)
\]

it results

\[
\mathbf{\Lambda^+ + \Lambda^-} = \mathbf{A}, \quad (2.91)
\]

\[
\mathbf{\Lambda^+ - \Lambda^-} = \mathbf{X}|\mathbf{\Lambda}|\mathbf{X^{-1}} = |\mathbf{A}|. \quad (2.92)
\]
And the CIR scheme can be written then in vector form, either in terms of characteristic variables

$$V^{n+1}_i = V^n_i - \frac{\Delta t}{\Delta x} \Lambda^+(V^n_i - V^n_{i-1}) - \frac{\Delta t}{\Delta x} \Lambda^-(V^n_{i+1} - V^n_i),$$  (2.93)

or -by means of the matrix $X$ - in terms of the starting variables

$$U^{n+1}_i = U^n_i - \frac{\Delta t}{\Delta x} A^+(U^n_i - U^n_{i-1}) - \frac{\Delta t}{\Delta x} A^-(U^n_{i+1} - U^n_i).$$  (2.94)

Techniques of flux splitting use expressions like (2.94).

**Godunov Method.** The linear hyperbolic system (2.68), is considered again, now written in conservative form

$$U_t + F_x = 0, \quad F(U) = AU.$$  (2.95)

The Godunov first order upwind scheme uses the conservative formula analogous to (2.58)

$$U^{n+1}_i = U^n_i - \frac{\Delta t}{\Delta x} \left( F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}} \right),$$  (2.96)

where, similarly to (2.61), the flux terms of the cell borders are

$$F_{i\pm\frac{1}{2}} = F\left(U_{i\pm\frac{1}{2}}(0)\right),$$  (2.97)

being $U_{i-\frac{1}{2}}(0)$ and $U_{i+\frac{1}{2}}(0)$ the solutions of $RP(U^n_{i-1}, U^n_i)$ and $RP(U^n_i, U^n_{i+1})$ respectively.

Numerical flows at the ends of the interval are calculated from the values of $F$ and $U$ in the anterior and posterior points yielding the expressions [27, pg. 185]

$$F_{i+\frac{1}{2}} = \frac{1}{2} (F^n_i + F^n_{i+1}) - \frac{1}{2} |A| \left(U^n_{i+1} - U^n_i\right),$$  (2.98)

$$F_{i-\frac{1}{2}} = \frac{1}{2} (F^n_{i-1} + F^n_i) - \frac{1}{2} |A| \left(U^n_i - U^n_{i-1}\right),$$  (2.99)

where $|A|$ takes the value given by (2.90).

That is, the resulting value for the flux vector at the left and right borders of the cell is the mean of the values at the points $(i - 1, i)$ and $(i, i + 1)$ respectively, with an upwinding term.

The $Q$-schemes use expressions like (2.98) and (2.99).
Chapter 3

Two-dimensional flow equations

3.1 Types of flow. Turbulent flow

The importance of the inertia forces with respect to the viscous ones in a particular flow is quantified by the dimensionless Reynolds number, which is calculated as the quotient between the two forces. Considering the magnitudes involved in both forces one obtains the usual expression of $Re$

$$Re = \frac{\rho V^2 L^2}{\mu VL} = \frac{VL}{\nu}. \quad (3.1)$$

$V$ and $L$ are the characteristic velocity and length of the flow and $\nu$ is the ratio between the dynamic viscosity and the density, called kinematic viscosity. It is observed experimentally that for values below the so-called critical Reynolds number the adjacent fluid layers slide over each other in an orderly way, which is called laminar regime. In a laminar regime, if the boundary conditions do not vary with time, the flux is permanent.

For values above critical $Re$, the flow behavior changes, becoming random and chaotic. The movement becomes non permanent, even with constant boundary conditions. It is called turbulent regime.

The random nature of turbulent flow and the high frequency with which the magnitudes vary make extremely difficult in practice a complete description of the movement of all fluid particles. Let $u, v, w$ the velocity components and $p$ the pressure. One can decompose a magnitude (for instance a velocity component $u$) in the sum of its average value ($\overline{u}$) and the turbulent fluctuation ($u'$). A turbulent flow is then characterized by the average values ($\overline{u}, \overline{v}, \overline{w}, \overline{p}$) and the statistical properties of the fluctuations ($u', v', w', p'$).

Even in flows where average velocities and pressures vary only in one or two spatial dimensions, the turbulent fluctuations are three-dimensional. If
we can visualize a turbulent flow we find fluid portions in rotation, called turbulent eddies. These have a wide spectrum of sizes being the largest eddies comparable to the dimensions of the domain. Inertial forces dominate in larger eddies, while its effect is negligible compared with the viscous forces in the smallest.

The energy required to maintain the motion of the larger eddies flow comes from the mean flow. On the other hand, smaller eddies obtained energy mainly from the higher ones. Thus kinetic energy is transmitted to increasingly smaller eddies through a cascade process, until it is dissipated by viscous forces. This dissipation causes the additional energy losses related to the turbulent flows.

### 3.2 Average value and fluctuation

As mentioned in the previous section, a magnitude \( \varphi \), which generally depends on time, can be decomposed into the sum of its average value plus a fluctuation component around this value.

\[
\varphi(t) = \overline{\varphi(t)} + \varphi'(t).
\] (3.2)

Although it does not appear explicitly in the expression of \( \varphi \) and \( \overline{\varphi} \), both are a function of the coordinates of the considered point \((x, y, z)\).

The temporal average of \( \varphi(t) \), for a given point, can be defined in different ways. For the cases considered here, of unsteady flow we will use the expression [21, pg. 278]

\[
\overline{\varphi(t)} = \frac{1}{\Delta t} \int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \varphi(\theta)d\theta,
\] (3.3)

where the chosen \( \Delta t \) must be greater than the time scale of the turbulence and lower than the time scale of the average flow. For example, in an estuary it can be considered that the period of the turbulent oscillation of the velocity is less than one second, while the tide period is about 12 hours. After performing the time average (3.3), the mean flow continues oscillating under the influence of the tide.

### 3.3 Navier-Stokes Equations

The shallow water equations are derived from the Navier-Stokes equations, which govern the behavior of a viscous fluid in three dimensions.
In incompressible fluids, density is independent of the pressure. In these fluids, mass per unit of volume can vary, for example due to temperature, but is considered constant with respect to the position and time.

Be a Cartesian system $x, y, z$, with $z$ positive upward. Calling $u, v, w$ the components of the velocity vector $\mathbf{u}$, these equations are expressed as:

**Continuity Equation** (conservation of mass)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$  \hfill (3.4)

**Dynamic Equation** (conservation of momentum)

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = F_x - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \Delta u,$$  \hfill (3.5)

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = F_y - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \Delta v,$$  \hfill (3.6)

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = F_z - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \Delta w.$$  \hfill (3.7)

The vector $\mathbf{F} = (F_x, F_y, F_z)^T$ is the force per unit mass; $p$ is the pressure; $\rho$ is the density; $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity, $\mu$ is the dynamic viscosity.

Adding to each of the three dynamic equations the continuity equation multiplied by $u, v$ and $w$ respectively, and using the $\nabla$ operator for divergence and $\nabla^2$ for the Laplacian, the system takes the form

$$\nabla \cdot \mathbf{u} = 0,$$  \hfill (3.8)

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{uu} = F_x - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u,$$  \hfill (3.9)

$$\frac{\partial v}{\partial t} + \nabla \cdot \mathbf{vu} = F_y - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v,$$  \hfill (3.10)

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{wu} = F_z - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 w.$$  \hfill (3.11)

### 3.4 Reynolds Equations in 3D

These equations govern turbulent movement of an incompressible fluid. To get them we replace, in the Navier-Stokes equations, the velocity and pressure by their time-averaged values plus the fluctuation terms.

$$\mathbf{u} = (u, v, w), \quad \mathbf{u} = \bar{u} + u', \quad v = \bar{v} + v', \quad w = \bar{w} + w', \quad p = \bar{p} + p'$$  \hfill (3.12)
CHAPTER 3. TWO-DIMENSIONAL FLOW EQUATIONS

and calculate the time average of each equation, obtaining

\[ \nabla \cdot \mathbf{u} = 0, \]  

\[ \frac{\partial \Pi}{\partial t} + \nabla \cdot \mathbf{u} \Pi + \nabla \cdot \mathbf{w} \mathbf{u}' = F_x - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 \mathbf{u}, \]  

\[ \frac{\partial \Pi}{\partial t} + \nabla \cdot \mathbf{u} \Pi + \nabla \cdot \mathbf{w} \mathbf{u}' = F_y - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 \mathbf{v}, \]  

\[ \frac{\partial \Pi}{\partial t} + \nabla \cdot \mathbf{w} \mathbf{u} + \nabla \cdot \mathbf{w} \mathbf{u}' = F_z - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 \mathbf{w}. \]

The resulting equations have the same structure as equations (3.8)-(3.11), with two differences: the variables \( u, v, w, p \) have been replaced by their average values and new ones have been added on the left member. If we develop them and place them on the right, we obtain the 3D Reynolds equations:

\[ \nabla \cdot \mathbf{u} = 0, \]  

\[ \frac{\partial \Pi}{\partial t} + \nabla \cdot \mathbf{u} \Pi + \nabla \cdot \mathbf{w} \mathbf{u}' = F_x - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 \mathbf{u} - \left[ \frac{\partial u^2}{\partial x} + \frac{\partial u' v'}{\partial y} + \frac{\partial u' w'}{\partial z} \right], \]  

\[ \frac{\partial \Pi}{\partial t} + \nabla \cdot \mathbf{v} \Pi + \nabla \cdot \mathbf{w} \mathbf{v}' = F_y - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 \mathbf{v} - \left[ \frac{\partial u' v'}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial v' w'}{\partial z} \right], \]  

\[ \frac{\partial \Pi}{\partial t} + \nabla \cdot \mathbf{w} \Pi + \nabla \cdot \mathbf{w} \mathbf{w}' = F_z - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 \mathbf{w} - \left[ \frac{\partial u' w'}{\partial x} + \frac{\partial v' w'}{\partial y} + \frac{\partial w^2}{\partial z} \right]. \]

The cross products of the turbulent fluctuations of velocity multiplied by density have dimensions of force/area and they are called Reynolds stresses. According to the Boussinesq hypothesis, these stresses are proportional to the derivative of the time averages of the velocity components, being \( \mu_t \), turbulent dynamic viscosity, the coefficient of proportionality. Operating, simplifying and assuming \( \nu + \nu_t \approx \nu_t \) we obtain:

\[ \nabla \cdot \mathbf{u} = 0, \]  

\[ \frac{\partial \Pi}{\partial t} + \nabla \cdot \mathbf{u} \Pi + \nabla \cdot \mathbf{v} \mathbf{u}' + \nabla \cdot \mathbf{w} \mathbf{u}' + \nu \nabla^2 \mathbf{u} = F_x - \frac{1}{\rho} \frac{\partial p}{\partial x} + \mathbf{v} \cdot \nu_t \frac{\partial \mathbf{u}}{\partial x}, \]  

\[ \frac{\partial \Pi}{\partial t} + \nabla \cdot \mathbf{v} \Pi + \nabla \cdot \mathbf{w} \mathbf{v}' + \nabla \cdot \mathbf{v} \mathbf{u}' + \nu \nabla^2 \mathbf{v} = F_y - \frac{1}{\rho} \frac{\partial p}{\partial y} + \mathbf{v} \cdot \nu_t \frac{\partial \mathbf{v}}{\partial y}, \]  

\[ \frac{\partial \Pi}{\partial t} + \nabla \cdot \mathbf{w} \Pi + \nabla \cdot \mathbf{w} \mathbf{w}' + \nabla \cdot \mathbf{w} \mathbf{u}' + \nu \nabla^2 \mathbf{w} = F_z - \frac{1}{\rho} \frac{\partial p}{\partial z} + \mathbf{w} \cdot \nu_t \frac{\partial \mathbf{w}}{\partial z}. \]

These expressions are very similar to the Navier Stokes Eq. (3.8)-(3.11) with the difference that the instantaneous values of velocity and pressure have
been replaced by their time averages, the viscosity by the turbulent viscosity \( \nu_t = \mu_t/\rho \) and a new addend has appeared in the source term, which is negligible if we assume that the spatial variation of \( \nu_t \) is very small.

3.5 The Shallow Water equations

There are flows in nature in which the horizontal dimensions are clearly predominant. If, in addition, the vertical variation in the horizontal velocity component is small and there are little vertical accelerations, these flows can often be described by means of a set of equations in two dimensions, resulting of the vertical integration of the 3D equations. To obtain them some hypothesis are made:

a) Small bottom slope.

b) The distribution of pressure is assumed to be hydrostatic.

c) The main movement of particles occurs in horizontal planes.

d) The vertical distribution of \( u, v \) is nearly uniform.

e) The mass forces are gravity and the Coriolis force.

f) The vertical acceleration of the particles is negligible compared to \( g \).

g) The contours friction in unsteady flow, can be evaluated using formulae valid for steady flow, like Manning.

The shallow water equations in two dimensions are expressed as

\[
\frac{\partial U}{\partial t} + \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = G, \tag{3.25}
\]

being the variables vector

\[
U = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \tag{3.26}
\]

terms \( F_1 \) and \( F_2 \)

\[
F_1 = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ hwv \end{pmatrix}, \quad F_2 = \begin{pmatrix} hv \\ hvw \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix}, \tag{3.27}
\]
and the source term

\[
G = \begin{pmatrix}
0 \\
fvh + \frac{\tau_s x}{\rho} + gh(S_{0x} - S_{fx}) + S_{t1} \\
-fuh + \frac{\tau_s y}{\rho} + gh(S_{0y} - S_{fy}) + S_{t2}
\end{pmatrix}.
\] (3.28)

The variable \( h \) represents the depth measured vertically, \( u, v \) are the averages in the vertical of the horizontal components of the velocity, \( f \) is the Coriolis coefficient, \( \tau_s \) evaluates the effect of wind, \( S_0 \) and \( S_F \) are the geometric and frictional slopes.

\[
S_{0x} = -\frac{\partial z_b}{\partial x}, \quad S_{0y} = -\frac{\partial z_b}{\partial y},
\] (3.29)

\[
S_{fx} = \frac{n^2 u \sqrt{u^2 + v^2}}{h^{4/3}}, \quad S_{fy} = \frac{n^2 v \sqrt{u^2 + v^2}}{h^{4/3}}.
\] (3.30)

The two components of the turbulent term \( S_t \) have the following expressions

\[
S_{t1} = \frac{\partial}{\partial x} \left( 2\nu_t h \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \nu_t h \left[ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right] \right),
\] (3.31)

\[
S_{t2} = \frac{\partial}{\partial x} \left( \nu_t h \left[ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right] \right) + \frac{\partial}{\partial y} \left( 2\nu_t h \frac{\partial v}{\partial y} \right).
\] (3.32)

Expressions (3.25)-(3.28) are known as the Shallow Water Equations in conservative form. It is common not to take into account neither the Coriolis and wind stress terms nor the turbulent term.
Chapter 4

Application to the 2D SWE

4.1 Types of finite volumes

The use of the finite volume method in computational fluid dynamics is relatively recent. According to Hirsch [17, pg. 237] it was introduced by McDonald in 1971 and independently by Mc Cormack and Paullay in 1972 for solving the Euler equations in 2D. In 1973 Rizzi and Inouye extended it to three-dimensional flows. Eymard et al. [13] attribute their introduction, ten years earlier, to Tichonov and Samarskii for solving convection-diffusion equations. In [13, pg. 9] a long list of works devoted to the mathematical analysis of the method is mentioned.

To use this method we usually start from a previous discretization of the computational domain in elements, normally triangles or quadrangles, from which the new mesh or finite volume cells is built. In each of these volumes the integral form of the equations is obtained, simplified by applying the divergence theorem and discretized. The resulting expressions establish the exact conservation of relevant flow properties in each cell. The terms of the equations are replaced by approximations of the finite difference type, obtaining algebraic equations that are solved by an iterative process. We briefly describe below some of the most commonly used finite volumes [4, 15].

4.1.1 Cell-type finite volumes

The cell type (or cell-center) finite volumes [15, pg. 366] are the same initial grid cells and the values of the dependent variables are stored in the cell center (centers of quadrilaterals or barycenters of triangles). This method has the advantage of using the initial mesh and the disadvantage that the nodes to which the variables values are assigned -which represent the cell
values and are used in the discretization- do not coincide with the nodes of the original mesh.

4.1.2 Vertex-type finite volumes

The vertex type (or cell-vertex) finite volumes [15, pg. 368] use the vertices of the initial mesh as nodes of the finite volume mesh and the new cells are built around them. In contrast to the previous case, the initial mesh vertices are used and assigned the variables values in each finite volume. In this method the implementation of boundary conditions is simpler, since the value of the variables in the boundary nodes are known. The drawback is that a new mesh has to be built (dual mesh).

4.1.3 Edge-type finite volumes

This type of finite volumes, not usual in literature, is described in [31, pg. 87]. To obtain them we start from a mesh formed by triangles, each of which is divided into three, by joining each vertex with the barycentre. Then the subtriangles are joined in pairs so that each finite volume is formed by the two subtriangles with an edge of the initial mesh in common. The center of the finite volume is the midpoint of the edge. With this method, the angular points of the contour -that belong to the initial mesh- are not nodes of the finite volume mesh, what avoids two difficulties. The first is related to the velocity vector: since fluid do not passes through the walls, in nodes corresponding to solid boundary the velocity vector must be parallel to the boundary, which gives problems in angular points. Another difficulty is the calculation of the perpendicular to the boundary edges at such points, which have two perpendicular. On the other hand, the initial nodes are not used. To obtain the values in these nodes we must interpolate.

4.2 Description of the finite volumes used

The finite volumes used in this work are based on a triangular discretization of the domains so that the nodes of the triangular mesh are used as the nodes of the finite volume mesh.

For each node I, the barycenters of the triangles with I as a common vertex and the mid-points of the edges that meet at I are taken. The boundary $\Gamma_i$ of the cell $C_i$ is obtained by joining these points and $\Gamma_{ij} = \overline{AMB}$ represents the common part of $\Gamma_i$ and $\Gamma_j$. 
The outward normal vector to $\Gamma_{ij}$ is $\eta_{ij}$ and it can be different at $\overline{AM}$ and at $\overline{MB}$. The norm of $\eta_{ij}$, $\|\eta_{ij}\|$, is the length of the edge and $\tilde{\eta}_{ij}$ is the corresponding unit vector

$$\tilde{\eta}_{ij} = \frac{\eta_{ij}}{\|\eta_{ij}\|} = (\tilde{\alpha}_{ij}, \tilde{\beta}_{ij})^T.$$  

The subcell $T_{ij}$ is the union of triangles AMI and MBI (see Figure 4.1). Their areas are

$$A_{ij1} = \frac{\|\eta_{ij}^{AM}\|d_{ij}^{AM}}{2}, \quad A_{ij2} = \frac{\|\eta_{ij}^{MB}\|d_{ij}^{MB}}{2},$$

where the values $d_{ij}$ represent the triangle height.

![Figure 4.1: Construction of the finite volumes.](image)

**4.3 Terms considered in the equations**

To apply the finite volume method to the Shallow Water Equations, first one has to decide which parts of the source term will be taken into account. In this regard, 1) the Coriolis term has almost no importance if the domain is not large; 2) it is usually neglected the influence of the wind, although in some cases, such as estuaries of some magnitude, it may be advisable to consider
it; and 3) the turbulent term is frequently not taken into consideration, assuming that its effects are included in the bottom friction term.

Removing the mentioned addends from the source term the equations (3.25)-(3.28) reduce to

\[
\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}_1 + \nabla \cdot \mathbf{F}_2 = \mathbf{G},
\]

(4.3)

being

\[
\mathbf{U} = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \quad \mathbf{F}_1 = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{pmatrix}, \quad \mathbf{F}_2 = \begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix},
\]

(4.4)

\[
\mathbf{G} = \begin{pmatrix} 0 \\ gh(S_{0x} - S_{fx}) \\ gh(S_{0y} - S_{fy}) \end{pmatrix}.
\]

(4.5)

The geometric slopes, defined in terms of \(z_b\) by (3.29) (see Fig. 4.2), will be expressed in terms of \(H\), which represents the distance to the bottom from a fixed reference level, positive downward. Thus the slopes are positive if the ground descend and negative otherwise.

\[
S_{0x} = \frac{\partial H}{\partial x}, \quad S_{0y} = \frac{\partial H}{\partial y}.
\]

(4.6)

The friction slopes are

\[
S_{fx} = \frac{n^2 u \sqrt{u^2 + v^2}}{h^{3/3}}, \quad S_{fy} = \frac{n^2 v \sqrt{u^2 + v^2}}{h^{4/3}},
\]

(4.7)

where \(n\) is the Manning coefficient, \(h\) the depth and \(u, v\) the horizontal components of the velocity. The conservative variables \(hu, hv\) represent the discharge per unit width in a rectangular channel. In Figure 4.2 a section of the domain by a vertical plane \(y = y_0\) is represented.

### 4.4 Integration and discretization

Using the simbolic notation

\[
\mathbf{F}(\mathbf{U}) = (\mathbf{F}_1(\mathbf{U}), \mathbf{F}_2(\mathbf{U})),
\]

(4.8)

the system (4.3) can be written in a more compact form

\[
\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F} = \mathbf{G},
\]

(4.9)
being $\nabla$ the operator $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ and $\nabla \cdot \mathbf{F}$ the divergence of $\mathbf{F}$.

The computational domain is divided into a set of finite volumes $C_i$, defined in 4.2, and the surface integral is calculated in each of them, obtaining

$$\int\int_{C_i} \frac{\partial \mathbf{U}}{\partial t} \, dA + \int\int_{C_i} \nabla \cdot \mathbf{F} \, dA = \int\int_{C_i} \mathbf{G} \, dA. \quad (4.10)$$

The divergence theorem is applied to the second term, turning the surface integral into a line integral along $\Gamma_i$, the cell border,

$$\int\int_{C_i} \frac{\partial \mathbf{U}}{\partial t} \, dA + \int_{\Gamma_i} \mathbf{F} \cdot \vec{\eta} \, dl = \int\int_{C_i} \mathbf{G} \, dA. \quad (4.11)$$

If the second term is moved to the other side and a minus sign is introduced in the integral, it results

$$\int\int_{C_i} \frac{\partial \mathbf{U}}{\partial t} \, dA = \int_{\Gamma_i} \mathbf{F} \cdot (-\vec{\eta}) \, dl + \int\int_{C_i} \mathbf{G} \, dA. \quad (4.12)$$

This equation can be interpreted saying that in an arbitrary domain (in particular, in each finite volume), the rate of variation of the conservative variables contained in $\mathbf{U}$ bar is given by minus the integral form of the flux of $\mathbf{F}$ normal to the boundary of the domain plus the “amount” of $\mathbf{U}$ generated
CHAPTER 4. APPLICATION TO THE 2D SWE

in the domain [22, pg. 6.20]. That is, the variation in time of $U$ is due to the net flux of $\mathbf{F}$ towards the inside of the cell plus the variation due to the source term.

4.4.1 Discretization of the time derivative

The solution of equation (4.3) is approximated by some values $U^n_i$, constant in each cell $C_i$ and time $t_n$, which are assigned to node $I$ corresponding to the cell.

The time derivative is discretized by the Forward Euler Method

$$\left. \frac{\partial U}{\partial t} \right|_{C_i,t_n} \approx \frac{U^{n+1}_i - U^n_i}{\Delta t}. \quad (4.13)$$

Then equation (4.11) becomes

$$\int\int_{C_i} U^{n+1}_i - U^n_i \frac{dA}{\Delta t} + \int_{\Gamma_i} \mathbf{F} \cdot \tilde{\mathbf{\eta}} dl = \int\int_{C_i} G dA. \quad (4.14)$$

In the first term, $U^{n+1}_i$, $U^n_i$ and $\Delta t$ have constant values in the cell, so they can go out of the integral, resulting

$$\int\int_{C_i} \frac{U^{n+1}_i - U^n_i}{\Delta t} dA = \frac{U^{n+1}_i - U^n_i}{\Delta t} A_i. \quad (4.15)$$

4.4.2 Integration of the flux and source terms

In the second addend of (4.14) the line integral splits in a sum of integrals along each of the edges $\Gamma_{ij}$, $j \in K_i$

$$\int_{\Gamma_i} \mathbf{F} \cdot \tilde{\mathbf{\eta}} dl = \sum_{j \in K_i} \int_{\Gamma_{ij}} \mathbf{F} \cdot \tilde{\mathbf{\eta}} dl. \quad (4.16)$$

The surface integral of the source term splits into a sum of integrals over the subcells $T_{ij}$, $j \in K_i$

$$\int\int_{C_i} G dA = \sum_{j \in K_i} \int\int_{T_{ij}} G dA. \quad (4.17)$$

Thus (4.14) becomes

$$\frac{U^{n+1}_i - U^n_i}{\Delta t} A_i + \sum_{j \in K_i} \int_{\Gamma_{ij}} \mathbf{F} \cdot \tilde{\mathbf{\eta}} dl = \sum_{j \in K_i} \int\int_{T_{ij}} G dA. \quad (4.18)$$
4.4.3 Definition of the discretized flux

The sum (4.16) will now be expressed in terms of the values of the variables at node \( I \) and \( N_j, \ j \in K_i \), using an upwind scheme.

The dot product of \( \mathbf{F} \) and \( \tilde{\eta} \) is called 2D flux through a segment of unit length

\[
Z = \mathbf{F} \cdot \tilde{\eta} = \tilde{\alpha} F_1 + \tilde{\beta} F_2,
\]

where \((\tilde{\alpha}, \tilde{\beta})\) are the components of the unit vector normal to the edge.

To discretize the flux different proposals have been made. In this case, we will use the Van Leer \( Q \)-scheme [30], as proposed in [5, 31].

The \( Q \)-schemes are a family of upwind schemes, in which the numerical flow is obtained as follows

\[
\phi(U^n_i, U^n_j, \tilde{\eta}_{ij}) = \frac{Z(U^n_i, \tilde{\eta}_{ij}) + Z(U^n_j, \tilde{\eta}_{ij})}{2} - \frac{1}{2} |Q(U^n_Q, \tilde{\eta}_{ij})| (U^n_j - U^n_i),
\]

where \( U^n_i, U^n_j \) represent the values of the variables vector at \( I \) and \( J \); \( Q \) is the jacobian matrix of flux

\[
Q = \frac{dZ}{dU} = \tilde{\alpha} \frac{dF_1}{dU} + \tilde{\beta} \frac{dF_2}{dU}.
\]

Matrix \(|Q|\) is obtained as

\[
|Q| = X |\Lambda| X^{-1},
\]

being \(|\Lambda|\) the diagonal matrix of the absolute values of the eigenvectors of \( Q \) and \( X \) the matrix whose columns are the eigenvectors of \( Q \); in the Van Leer \( Q \)-scheme, \(|Q|\) is evaluated in the intermediate state

\[
U^n_Q = \frac{U^n_i + U^n_j}{2}.
\]

Then, expression (4.20) that discretizes the 2D flux in the middle point between \( I \) and \( J \) is obtained as the mean value of the fluxes at both points plus an upwinding term.

Note that in (4.21), the derivatives of \( F_1 \) and \( F_2 \) with respect to the variables vector \( U \) are the factors that appear when writing the equation (4.3) in the nonconservative form.

\[
\frac{\partial U}{\partial t} + \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \frac{\partial U}{\partial t} + \frac{dF_1}{dU} \frac{\partial U}{\partial x} + \frac{dF_2}{dU} \frac{\partial U}{\partial y} = G.
\]
The shallow water equations are a strictly hyperbolic system (see 2.7), i.e. the flux Jacobian matrix has three different eigenvalues and three linearly independent eigenvectors, so there will always be $X^{-1}$. Indeed

$$\frac{dF_1}{dU} = \begin{pmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -uv & v & u \end{pmatrix}, \quad (4.25)$$

$$\frac{dF_2}{dU} = \begin{pmatrix} 0 & 0 & 1 \\ -uv & v & u \\ -v^2 + gh & 0 & 2v \end{pmatrix}, \quad (4.26)$$

therefore, applying (4.21), it results

$$\frac{dZ}{dU} = \begin{pmatrix} 0 \\ \tilde{\alpha}(-u^2 + gh) + \tilde{\beta}(-uv) \\ \tilde{\alpha}(-uv) + \tilde{\beta}(-v^2 + gh) \end{pmatrix} \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\alpha}u + \tilde{\beta}v & \tilde{\beta}u \end{pmatrix}. \quad (4.27)$$

If $c$ is the speed of the wave

$$c = \sqrt{gh}, \quad (4.28)$$

the eigenvalues of the Jacobian matrix are

$$\lambda_1 = \tilde{\alpha}u + \tilde{\beta}v,$$
$$\lambda_2 = \lambda_1 + c,$$
$$\lambda_3 = \lambda_1 - c, \quad (4.29)$$

and matrices $|\Lambda|$, $X$ and $X^{-1}$ can be calculated

$$|\Lambda| = \begin{pmatrix} |\lambda_1| & 0 & 0 \\ 0 & |\lambda_2| & 0 \\ 0 & 0 & |\lambda_3| \end{pmatrix}, \quad (4.30)$$

$$X = \begin{pmatrix} 0 & 1 & 1 \\ -\tilde{\beta}c & u + \tilde{\alpha}c & u - \tilde{\alpha}c \\ \tilde{\alpha}c & v + \tilde{\beta}c & v - \tilde{\beta}c \end{pmatrix}, \quad (4.31)$$

$$X^{-1} = \frac{1}{2c} \begin{pmatrix} 2\tilde{\beta}u - 2\tilde{\alpha}v & -2\tilde{\beta} & 2\tilde{\alpha} \\ c - \tilde{\alpha}u - \tilde{\beta}v & \tilde{\alpha} & \tilde{\beta} \\ c + \tilde{\alpha}u + \tilde{\beta}v & -\tilde{\alpha} & -\tilde{\beta} \end{pmatrix}. \quad (4.32)$$
Thus, the second member of (4.16) is discretized as
\[
\sum_{j \in K_i} \int_{\Gamma_{ij}} \mathbf{F} \cdot \mathbf{\tilde{\eta}} \, dl \approx \sum_{j \in K_i} \| \mathbf{\eta}_{ij} \| \mathbf{\phi}_{ij}^n, \tag{4.33}
\]
being \( \mathbf{\phi}_{ij}^n \) the discretized unit flux
\[
\mathbf{\phi}_{ij}^n = \frac{\mathbf{\tilde{\alpha}} \mathbf{F}_1 + \mathbf{\tilde{\beta}} \mathbf{F}_2}{2} + \frac{\mathbf{\tilde{\alpha}} \mathbf{F}_1 + \mathbf{\tilde{\beta}} \mathbf{F}_2}{2} - \frac{1}{2} (\mathbf{X} |\mathbf{\Lambda}| \mathbf{X}^{-1})_{ij} (\mathbf{U}_j - \mathbf{U}_i). \tag{4.34}
\]

4.4.4 Definition of the discretized source

The convenience of the upwinding of the source term has been analyzed by Vázquez Cendón [31]. In this work she studies the discretization of the geometric source term in the 1D shallow water equations and proposes, for the 2D case, an extension of the 1D expression, verifying that gives good results. Here we will upwind the source term containing the geometric slope and calculate the friction slope in the center of each cell. According to the above, the two-dimensional discretized source in each subcell \( T_{ij} \) is defined as
\[
\psi = \mathbf{X} (\mathbf{I} - |\mathbf{\Lambda}| \mathbf{\Lambda}^{-1}) \mathbf{X}^{-1} \mathbf{\hat{G}}_0 + \mathbf{\hat{G}}_f, \tag{4.35}
\]
where \( \psi = (1 - |Q| Q^{-1}) \mathbf{\hat{G}}_0 + \mathbf{\hat{G}}_f \) where \( \mathbf{X}, \mathbf{X}^{-1}, |\Lambda| \) and \( \Lambda^{-1} \) are calculated at \( (\mathbf{U}_Q, \mathbf{\tilde{\eta}}_{ij}) \), Matrices \( |\mathbf{\Lambda}|, \mathbf{X} \) y \( \mathbf{X}^{-1} \) are given respectively by (4.30), (4.31) and (4.32). \( \Lambda^{-1} \) is
\[
\Lambda^{-1} = \begin{pmatrix}
1/\lambda_1 & 0 & 0 \\
0 & 1/\lambda_2 & 0 \\
0 & 0 & 1/\lambda_3
\end{pmatrix}, \tag{4.36}
\]

\( \mathbf{\hat{G}}_0 \) approximates the geometric source term
\[
\mathbf{\hat{G}}_0 = \begin{pmatrix}
0 \\
\frac{h_i^n + h_j^n}{2} \frac{H_j - H_i}{d_{ij}} \mathbf{\tilde{\alpha}} \\
\frac{h_i^n + h_j^n}{2} \frac{H_j - H_i}{d_{ij}} \mathbf{\tilde{\beta}}
\end{pmatrix}, \tag{4.37}
\]
and \( \mathbf{\hat{G}}_f \) approximates the friction source term, evaluated in each cell center.
\[
\mathbf{\hat{G}}_f = \begin{pmatrix}
0 \\
gh_i^n (-S_{fx})_i^n \\
gh_i^n (-S_{fy})_i^n
\end{pmatrix}. \tag{4.38}
\]
CHAPTER 4. APPLICATION TO THE 2D SWE

$(\tilde{\alpha}, \tilde{\beta})$ and $d_{ij}$ take the values given by expressions (4.1) and (4.2) and the friction slopes are given by (4.7). The sum of (4.17) is then expressed as

$$\sum_{j \in \mathcal{K}_i} \int_{T_{ij}} G \, dA \approx \sum_{j \in \mathcal{K}_i} A_{ij} \psi_{ij},$$

(4.39)

where $A_{ij}$ takes the expressions given in (4.2) for each subtriangle of the subcell $T_{ij}$ and $\psi_{ij}$ is calculated from (4.35). Note that $d_{ij}$ represents the distance from $I$ to the edge $AM$.

4.4.5 Discretization of the boundary conditions

To discretize the flux term in the contour of the cell and the source term in each subcell we use the variables values at $I$ and its neighbor $J$. In the case of the boundary nodes, there are no neighboring nodes on the other side of the edge (Figure 4.1). We assume, for these edges, that the neighboring node is the same node $I$, which means that we do not upwind the flux (as the upwinding term is zero). Furthermore, the area of each subcell is a factor of the discretized source. As edges have zero area, we do not consider the source terms in these nodes.

4.4.6 Obtaining of the time step

There are different expressions for calculating the time step that ensures stability. For the one dimensional case, taking a maximum value of Courant number equal to 1, the condition used in [18, pg. 283] is

$$\Delta t \leq \frac{\min (\Delta x)_i}{\max (|u| + c)_i},$$

(4.40)

being $(\Delta x)_i$ the cell width, $u$ the velocity and $c$ the celerity.

For the 2D case, [1, pg. 233] proposes

$$\Delta t \leq \min \left( \frac{D_{ij}}{2(\sqrt{u^2 + v^2} + c)_{ij}} \right),$$

(4.41)

being $D_{ij}$ the distances between node $I$ and its four neighbors. This has been the formula used in the present work.
4.5 Algorithm

A forward discretization of the time derivative and two others for the flux and source terms, which are evaluated at time \(t_n\) have been obtained. Thus (4.18) now takes the form

\[
\frac{U_i^{n+1} - U_i^n}{\Delta t} A_i + \sum_{j \in K_i} \|\eta_{ij}\| \phi_{ij}^n = \sum_{j \in K_i} A_{ij} \psi_{ij}^n, \quad (4.42)
\]

from where

\[
U_i^{n+1} = U_i^n + \frac{\Delta t}{A_i} \left( \sum_{j \in K_i} A_{ij} \psi_{ij}^n - \sum_{j \in K_i} \|\eta_{ij}\| \phi_{ij}^n \right). \quad (4.43)
\]

The above expression provides a explicit in time iterative method for calculating the value of the variables vector \(U\) on each node \(I\) and in every moment, from the variables values in the previous time, on the same node \(I\) and the \(N_j, j \in K_i\) that surround it.
Chapter 5

Some results

5.1 Types of boundary conditions

The conditions usually used for channel flow are of Dirichlet type: the values of some variables at certain points are set.

- Side walls: In channels where we simulate unidimensional flow, we impose the slip-condition, i.e. we cancel the velocity component perpendicular to the wall. This means considering section wide enough to neglect the friction on the walls. In the two-dimensional flow, the wall friction is considered by changing the value of the hydraulic radius of the nodes in contact with it.

5.2 One dimensional problems

Both the one dimensional and the two dimensional examples have been calculated with the two dimensional model. In the one dimensional ones a symmetric channel with a symmetric mesh has been considered with the slip-condition (no friction) as the boundary condition in walls. In these cases, we represent the depth along a longitudinal section.
5.2.1 Straight channel with slope and bottom friction

The calculations have been performed for a channel 2 m wide by 1,000 m long. The mesh is formed by right angled triangles whose arms are 1 m long with a total of 3003 nodes. The unit discharge is \( q = 4 \text{ m}^2/\text{s} \), and the critical depth is

\[
y_c = \left( \frac{q^2}{g} \right)^{1/3} = 1.18 \text{ m}.
\]  

(5.1)

Two flows, one subcritical and another supercritical have been simulated.

a) Subcritical flow: The slope is \( S_0 = 0.001 \) and the Manning coefficient \( n = 0.015 \) (concrete) [10, pg. 109]. The normal depth is

\[
y_n = \frac{n^{0.6} q^{0.6}}{S_0^{0.3}} = 1.47 \text{ m},
\]  

(5.2)

greater than \( y_c \). As the flow is subcritical, the imposed boundary conditions are: discharge at the inlet section and depth at the outlet. Two cases are solved: the first one with an imposed depth of \( h_{bc} = 1.75 \text{ m} \), greater than \( y_n \), the second one with \( h_{bc} = 1.20 \text{ m} < y_n \). In both cases the initial conditions are still water and constant depth equal to the \( h_{bc} \). The flow profile (of type M1 and M2, respectively) can be seen in figures 5.1 and 5.2. In both cases the depth gradually approaches its normal value.

b) Supercritical flow: The values are \( S_0 = 0.002 \) and \( n = 0.01 \) (glass). In this case the normal depth results

\[
y_n = \frac{n^{0.6} q^{0.6}}{S_0^{0.3}} = 0.94 \text{ m},
\]  

(5.3)

smaller than \( y_c \). The boundary conditions (supercritical flow) are imposed at the inlet. In the first case \( h_{bc} = 1.15 \text{ m} > y_n \) and in the second \( h_{bc} = 0.70 \text{ m} < y_n \). The initial conditions are the same as above. The flow profiles are of types S2 and S3, (figures 5.3 and 5.4 respectively). Again the depth gradually approaches its normal value in both cases.
**CHAPTER 5. SOME RESULTS**

Figure 5.1: Subcritical flow (M1). Downstream depth $h > h_n$.

Figure 5.2: Subcritical flow (M2). Downstream depth $h < h_n$. 
### Figure 5.3: Supercritical flow (S2). Upstream depth $h > h_n$. 

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### Figure 5.4: Supercritical flow (S3). Upstream depth $h < h_n$. 

---
5.2.2 Channel with an obstacle on the bottom

We study now the flow over an obstacle. The channel is 25 m long and 1 m wide, with a parabolic obstacle in the bottom. The mesh is formed by right angled triangles whose arms are 0.25 m long with a total of 505 nodes. The bottom levels are given by

\[
z(x) = \begin{cases} 
0.2 - 0.05(x - 10)^2 & \text{si } 8 < x < 12, \\
0 & \text{en el resto del dominio.}
\end{cases}
\]  

(5.4)

The Manning coefficient is \( n = 0.01 \). Three cases are shown:

- **Subcritical flow.** The unit discharge is \( q = 4.42 \, m^2/s \) and the downstream fixed depth \( h_{bc} = 2 \, m \). The flow is subcritical and there must be a decrease in the fluid depth when passing over the bump [9, pg. 35]. The computational result is represented in Figure 5.5.

- **Transcritical flow with hydraulic jump.** The unit discharge is \( q = 0.18 \, m^2/s \) and the downstream fixed depth \( 0.33 \, m \). A flow transition is produced over the obstacle (from sub to supercritical); then, due to the downstream fixed depth (greater than \( y_c \)) there is a second flow transition (from super to subcritical) through a hydraulic jump. The result can be seen in Figure 5.6.

- **Transcritical flow without hydraulic jump.** The unit discharge is \( q = 1.53 \, m^2/s \) and the downstream depth is set at 0.66 m, only while the flow is subcritical. As in the previous case, the flow is subcritical at the beginning. Over the obstacle a flow transition is produced but, by eliminating the downstream depth, the flow does not change to subcritical. The result is shown in Figure 5.7.

The initial conditions are the same in the three cases: still water and horizontal flow profile corresponding to the downstream depth \( h_{bc} \). The results obtained in the tests coincide with those of [32].
Figure 5.5: Subcritical flow over a bump.

Figure 5.6: Transcritical flow with hydraulic jump.
5.2.3 Dam break

In this section we simulate the progression of a wavefront in the so-called Dam Break problem. The fluid is still with different depths on both sides of one imaginary gate. At time $t = 0$ the gate is removed and the fluid is allowed to evolve freely. The slip-condition is imposed on all the boundary.

The domain is a channel $4 \times 200 \text{ m}^2$. The triangles that form the mesh arms $1 \text{ m}$ long with a total of $1,005$ nodes. The bottom is flat and without friction. The gate is located at the midpoint of the channel. The calculation is performed with dry and wet bottom, i.e. with the following initial conditions:

- Wet bottom
  - Upstream depth: $H_1 = 1 \text{ m}$.
  - Downstream depth: $H_0 = 0.1 \text{ m}$.

- Dry bottom
  - Upstream depth: $H_1 = 1 \text{ m}$.
  - Downstream depth: $H_0 = 0.0 \text{ m}$.

In the first case, the depths are represented in the instant $t = 25 \text{ s}$. The model captures well the discontinuity that the front represents. In the second
problem, with dry bottom, the depths at time $t = 15$ s are shown in Figure 5.9.

In both cases the result is a good approximation of the exact solution for 1D, obtained from [28]. The analytical solutions can be found in [31, pg. 60] and its obtaining in [25, pgs. 308 y 333].

5.2.4 Dam break with reflection

The last one dimensional problem is another case of Dam Break, in a channel 1 x 50 m$^2$, also with flat bottom and no friction. The arms of the triangles of the mesh are 0.25 m long with a total of 1,005 nodes. The gate is located at the midpoint of the channel. The boundary conditions are slip-condition on the sidewalls and zero discharge in the first and last channel sections.

After eliminating the gate, the wave begins to move on dry bed. Upon reaching the limit of the domain, a reflection takes place and the front starts moving in the opposite direction, now over wet bed. This going and coming is repeated again and again, in each cycle decreasing the height of the front. After 3140 seconds the representation of the free surface is a horizontal line with the scale used in Figure 5.11.
Figure 5.8: Dam break over wet bed. $t = 25 \text{ s}$. 

Figure 5.9: Dam break over dry bed. $t = 15 \text{ s}$. 
CHAPTER 5. SOME RESULTS

Figure 5.10: Dam Break with reflection 1.

Figure 5.11: Dam Break with reflection 2.
5.3 Two dimensional problems

5.3.1 Partial Dam Break

Now a two-dimensional Dam Break problem is solved. The domain is now square divided into two parts by a wall. The gap in the wall is partial and asymmetric. The domain is 200 x $m^2$ with nodes every 5 m, with 1656 computational nodes. There is no friction on the bottom and walls.

Two cases are solved with the following initial conditions of depth:

- Depth: upstream $H_1 = 10$ m and downstream $H_0 = 5$ m.
- Depth: upstream $H_1 = 5$ m and downstream $H_0 = 0$ m.

In Figure 5.12 a wavefront can be seen that, after a time $t = 7.1$ s, reaches the right side of the contour. Figure 5.13 represents the velocity field. The wavefront in Figure 5.14 is clearly different, due to dry bed condition. Finally (Figure 5.15) shows the mesh used in both cases.

Figure 5.12: 2D Dam Break. $H_1/H_0 = 2$. $t = 7.1$ s.
Figure 5.13: 2D Dam Break. $H_1/H_0 = 2$. Velocities. $t = 7.1$ s.

Figure 5.14: 2D Dam Break. $H_0 = 0$. $t = 7.1$ s.
Figure 5.15: 2D Dam Break. Computational mesh.

Figure 5.16: Fishway. Computational mesh.
5.3.2 Flow in a fishway

A dam in a river is a barrier to the free movement of fish, which is particularly important in the case of species that need to go upstream to spawn. To solve this problem fishways can be used, but their slope is generally high, producing an excessive water velocity. It can be reduced by means of deflectors that generate high energy dissipation, creating areas of low speed, where the fish can rest during the climb.

In the Center for Technology Innovation in Building and Civil Engineering (CITEEC), University of A Coruña, measurements were made of depths and velocities with two fishway designs for different slopes and discharges [26]. The experimental setup consists of nine pools, four of each type with an intermediate of transition. The two types are shown in Figures 5.17 and 5.18 and, in the first one, also the position of the measurement points. The experimental measurements for the different slopes and discharges analyzed, have been made in the pool 7, the third from the bottom. This is why the computational domain is a set of three pools, numbered 1 through 3 in the direction of flow, to compare the values of the variables in the number 1 with the experimental results.

The calculations have been performed for a slope of 5.7% and a discharge of 24.6 l/s. The Manning coefficient is 0.015. Despite the steep slope and because of the action of the deflectors, the flow is subcritical. The boundary condition of depth is imposed on the outlet section of pool 3, fixing in each point of the section the depth obtained from the measurements.

Figure 5.16 shows the mesh used. In Figures 5.19 to 5.21 velocity vectors, velocity modules and depth values are shown. In the six following figures a comparison is made between the computational values obtained for the first pool and the experimental values.
Figure 5.17: Fishway. Type I. Distances in $cm$.

Figure 5.18: Fishway. Type II. Distances in $mm$. 
Figure 5.19: Fishway. Velocities.

Figure 5.20: Fishway. Velocity modulus.

Figure 5.21: Fishway. Depths.
Figure 5.22: Fishway, pool 1. Computational velocities.

Figure 5.23: Fishway, pool 1. Measured velocities.
Figure 5.24: Fishway, pool 1. Computational velocity modulus.

Figure 5.25: Fishway, pool 1. Measured velocity modulus.
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Figure 5.26: Fishway, pool 1. Computational depth.

Figure 5.27: Fishway, pool 1. Measured depth.
References


[6] Brigham Young University (1996), *User’s guide to RMA2 version 4.3* Salt Lake City, EEUU.


