An optimal strategy for maximizing the expected real-estate selling price: accept or reject an offer?

Martín Egozcue[1]

Facultad de Ciencias Sociales, Universidad de la República, Uruguay, Montevideo 11200, Uruguay, and

Accounting and Finance Department, Norte Construcciones, Punta del Este, Maldonado 20100, Uruguay

Luis Fuentes García

Departamento de Métodos Matemáticos e de Representación, Universidade da Coruña, 15001 A Coruña, Spain

Ričardas Zitikis

Department of Statistical and Actuarial Sciences, University of Western Ontario, London, Ontario N6A 5B7, Canada

Abstract. Motivated by a real-life situation, we put forward a model and then derive an optimal strategy that maximizes the expected real-estate selling price when one of the only two remaining buyers has already made an offer but the other one is yet to make. Since the seller is not sure whether the other buyer would make a lower or higher offer, and given no recall, the seller needs a strategy to decide whether to accept or reject the first-come offer. The herein derived optimal seller's strategy, which maximizes the expected selling price, is illustrated under several scenarios, such as independent and dependent offers by the two buyers, and for several parametric price distributions.

[1]Corresponding author. E-mail: egozcuemj@gmail.com

Keywords and phrases: Real estate; Optimal strategy; Decision theory; Uncertainty; Two-envelope problem.

1 Introduction

1.1 The motivating problem

The following problem has been posited by a real estate brokerage to the authors of this paper.

The seller of a house – whether on his own or with the help of a real estate agent, or perhaps both (cf., e.g., Salant 1991) – sets a list price of the house and requests the sales agent to proceed with the sale. We refer to, e.g., Zorn and Larsen (1986), Miceli (1989), Larsen and Park (1989), and Jares et al. (2000), and references therein, on how to provide incentives for real estate sales agents to act in the best interests of home sellers.

A number of buyers, some serious and others just curious, view the house, and perhaps even make exploratory offers, which the seller can use to revise his reservation price as explained by, e.g., Read (1988). After some time, the sales agent tells the seller that the matter has reached the stage when there are left only two serious buyers. Naturally, some bargaining would take place.

Denote the two buyers by the first letters of their (fictional) names, *L* and *H*, and we do not know, though perhaps attempt to guess, which of the two would be the first to make an offer. Let X_L and X_H be the (random) sale prices if the house is to be sold to *L* or *H*, respectively. After realizing one of the prices, either X_L or X_H but we do not know which of the two, the seller needs to decide whether to accept the first-come offer or reject it and then bargain with the remaining buyer.

The seller is aware of the fact that if he rejects the first-come offer, then the first-come buyer would exit the process due to reasons such as buying another house, or simply because of getting his ego hurt, as is quite often the case in such situations. Hence, there is no recall, and thus if rejected, the buyer exits the process and leaves the seller with only one buyer, whose offered price, perhaps after some bargaining, would be accepted as the final selling price.

The seller wants to have a selling strategy, which needs to be determined prior to acting on the first-come offer. The need for such a strategy arises because the seller feels, naturally, that one of the two offers is likely to be higher than the other one, but he does not know which of them – the higher or the lower – will come first. Hence, accepting or rejecting the first-come offer is a crucial step for the seller, and the aim of the present paper is to offer an optimal strategy for the seller who wishes to maximize the expected selling price. We call the strategy *maximizing*.

Our research of the literature, especially of that concerned with strategies in the real estate business (Subsection 1.2 below), has revealed that the above formulated problem does not really fit into the models considered so far. Certainly, we have greatly benefited from the literature, but the closest solution to our problem has turned out to be related to problems, or puzzles, on the theme "which of the two numbers is larger when only one of them is shown to you?" Here we

ACCEPTED MANUSCRIPT

mention only two such problems: the two-envelope problem and the secretary problem, with more details and references to be provided later in this paper.

1.2 A glimpse of the real-estate literature

Optimal strategies for selling assets in general, and thus real estate in particular, have been actively studied in the literature (e.g., DeGroot 1970; Albright 1977; Riley and Zeckhauser 1983; Rosenfield et al. 1983). Some works assume that the seller receives a sequence of random bids arriving in a stochastic manner. Some assume that rejected offers are not lost (recall), and others that they are lost (no recall). Some assume that the distribution of offers is known, and others that it is not. It is quite often assumed that the bids are independent and identically distributed random variables. A number of authors derive stopping-type rules that lead to best strategies for selling assets. Rosenfield et al. (1983) provide a list of selling strategies within various frameworks.

Building upon, and extending, several earlier works (e.g., Stigler 1961; Nelson 1970) on the economics of information, Gastwirth (1976) has investigated the problem of consumer search for information about price and quality of goods. He explores a sequential procedure as a search strategy, which essentially suggests searching until a price below a threshold has been found. Gastwirth (1976) explores the effects of various distributions of prices (with bounded and unbounded supports) on the search length, as well as the effects of possible dependencies between the prices. Deng et al. (2013) analyze the reservation and asking prices, putting an

ACCEPTED MANUSCRIPT

emphasis – articulated already by Stigler (1961) – on the dispersion of prices and also investigating, among other things, the influence of the dispersion on pricing strategies that maximize the return from search.

Naturally, the distribution of prices plays a pivotal role. It has been shown, for example, that the upper bound of the distribution support may coincide with the listing price, but it may also exceed it. The list price can be determined by the seller himself or with the help of a broker (cf., e.g., Salant 1991). The price can also be predicted – with some success – by inflating previous house selling prices (e.g., Brint 2009).

It should be noted that setting the `right' list price is a complex task and plays a pivotal role in determining factors such as the time on the market (known as TOM) and the price of the property. These effects of the list price have been explored theoretically as well as using empirical evidence by Yavas and Yang (1995), Arnold (1999), Anglin et al. (2003), among others. In the case of a sequential search with recall, Cheng et al. (2008) have derived a closed form formula of the TOM and the price, and they have shown in particular that the two quantities follow a nonlinear positive relationship.

Sirmans et al. (1995) have examined the prices of quickly selling houses. Their model assumes that bids are independent and identically distributed random variables. Following Lippman and McCall (1976), who suggest and explore a model of job search based on wage amounts, Sirmans et al. (1995) use a stopping rule as the seller's strategy: accept the first bid if it is larger than the reservation price and reject it otherwise. The authors analyze various quantities such as the optimal reservation price and how it is affected by holding costs and seller's information about the distribution of offers.

Glower et al. (1998) have studied how seller's motivation influences the selling time, list and final prices. In particular, they have investigated five factors that affect the seller's motivation: 1) the seller has a moving plan at the date of the price listing, 2) the seller has accepted a new employment prior to the time of the listing, 3) the seller has made an offer or bought another house at the time of the listing, 4) the sale is atypical, and 5) the seller has set an incorrect price. The model of Glower et al. (1998) is without recall, and the number of received offers is not limited.

Related to the `motivational paradigm,' Anglin (2004) has studied optimal strategies for households that must sell one house in order to buy another. Arnold (1999) has derived optimal asking and reservation prices. The model of Arnold (1999) assumes that the seller faces buyers arriving according to the Poisson process with some intensity, and that the asking price, which serves as a starting price in the bargaining process, affects the intensity of the arrival of potential buyers. Biswas and McHardy (2007) have studied fixed-price and asking-price strategies for selling assets in uncertain markets, as well as the determination of associated price discounts.

Naturally, earlier derived selling strategies as well as those to be explored later in this paper hinge on price distributions and other factors. A number of price distributions have been proposed in the literature. For example, Horowitz (1992) puts forward a theory of seller's behaviour, suggests a distribution of (random) bids, derives optimal list and reservation prices,

ACCEPTED MANUSCRIPT

and explains why there are list prices in the housing market and why bids can sometimes be above the corresponding list prices.

Bid or price distributions can be with finite or infinite supports. In Gastwirth (1976), for example, we find uniform, triangular, and normal distributions. We can also argue in favour of the lognormal distribution, but Ohnishi et al. (2011) explain why the heavier tailed Pareto distribution might be better. In Section 3 below, we shall use some of these distributions to illustrate our proposed optimal threshold-type strategies for selling real estate.

Certainly, we have not attempted to give here a general literature overview on the topic, which is vast and spans through numerous journals on real estate, decision theory, economics, operations research, management science, and other areas. However, we hope to have provided a glimpse of those aspects that have been discussed in the literature and – in one way or another – have profoundly influenced our thinking on, and the solution of, the motivating problem formulated above.

The rest of the paper is organized as follows. In Section 2, we put forward a probabilistic model that corresponds to our motivating example, and we also formulate natural and practically sound assumptions under which we derive a formula for the expected selling price. The formula leads to a maximizing seller's strategy in Section 3, where we explore two important cases in detail: 1) the (random) prices X_L and X_H are independent, though not necessarily identically distributed, and 2) the prices are tied by a dependency relationship. Similarly to the optimal sequential

ACCEPTED MANUSCRIPT

stoping strategies noted earlier, we shall see in Section 3 that in both cases 1) and 2), we also arrive at optimal threshold-type strategies: reject the first-come offer if it is below a certain threshold and accept it otherwise. Our developed theory provides a constructive definition of the threshold, which can thus be calculated or estimated in practice.

2 The model and the main theorem

As noted earlier, we do not know which of the two, *L* or *H*, will buy the house, as the outcome depends on factors such as who is going to offer first and at what price, and whether the seller accepts or rejects the first-come offer.

Under this uncertainty, we are interested in maximizing the expected selling price $\mu_X = \mathbf{E}[X]$ which naturally depends on a certain seller's strategy. We want to know this strategy.

We shall next introduce some fairly natural assumptions that will facilitate the tractability of the aforementioned maximization problem.

2.1 Main assumptions

Let O_1 be the random variable that takes on the two `categorical' values *L* and *H*: if $O_1 = L$, then the first-come offer is by the buyer *L*, but if $O_1 = H$, then the first-come offer is by *H*.

Next, let R_1 be the random variable of rejecting the first-come offer, that is, R_1 takes on the `categorical' value $Y$ (`yes') if the first-come offer is rejected, and on the value N (`no') otherwise.

Assumption 2.1 Whether the first-come offer O_1 is made by L or H does not depend on the (random) prices X_L and X_H.

From the mathematical point of view, Assumption 2.1 means that the conditional probability $\mathbf{P}[O_1 = L \mid X_L, X_H]$ is equal to the unconditional probability

$$p \equiv \mathbf{P}[O_1 = L]$$

of the first-come offer by L. Consequently, the probability $\mathbf{P}[O_1 = H \mid X_L, X_H]$ of the first-come offer by H given the prices X_L and X_H is equal to the unconditional probability $\mathbf{P}[O_1 = H]$; the latter is equal to $1 - p$.

Assumption 2.2 The probability of rejecting the first-come offer depends *only* on the amount that the first-come buyer offers.

Hence, for example, the probability $\mathbf{P}[R_1 = Y \mid X_L = x, X_H = y, O_1 = H]$, which can be rewritten as $\mathbf{P}[R_1 = Y \mid X_L = x, X_{O_1} = y, O_1 = H]$, is equal to

$$\mathsf{S}(y) \equiv \mathbf{P}[R_1 = Y \mid X_{O_1} = y].$$

We call S the seller's strategy function, or simply the seller's strategy. Hence, $\mathsf{S}(y)$ is the probability of rejecting the first-come offer of size y irrespectively of whether L or H makes the offer. Hence, in particular,

• when $\mathsf{S}(y) = 1$, then the first-come offer is rejected, and when $\mathsf{S}(y) = 0$, then it is accepted. (In this sense, we can view S as a `rejection strategy.')

Analogous arguments under Assumption 2.2 imply that $\mathbf{P}[R_1 = N \mid X_L = x, X_H = y, O_1 = L]$ is equal to $1 - \mathsf{S}(x)$.

Finally, we introduce the benchmark expected price (BEP)

$$\text{BEP} = p\mathbf{E}[X_L] + (1-p)\mathbf{E}[X_H],$$

which is the expected selling price if we were always to accept the first-come offer. Obviously, BEP is a `strategy-less' quantity. In the next subsection we shall look at the difference between μ_X and BEP, where the seller's strategy S will play a crucial role.

ACCEPTED MANUSCRIPT

2.2 The main theorem

Depending on the seller's strategy S, the expected selling price μ_X might be higher or lower than the strategy-less BEP. The following theorem specifies the strategy risk parameter (SRP), which is the difference between μ_X and BEP. The proof of the theorem is long and thus relegated to Appendix A at the end of this paper.

Theorem 2.1 The expected selling price μ_X is the sum of the (strategy-less) benchmark expected price BEP and the (strategy-dependent) strategy risk parameter

$$\mathrm{SRP}(\mathsf{S}) = p\mathbf{E}[\mathsf{S}(X_L)\{\mathbf{E}[X_H \mid X_L] - X_L\}] + (1-p)\mathbf{E}[\mathsf{S}(X_H)\{\mathbf{E}[X_L \mid X_H] - X_H\}],$$

that is, we have the decomposition $\mu_X = \mathrm{BEP} + \mathrm{SRP}(\mathsf{S})$.

With the help of Theorem 2.1, we can now aim at deriving a strategy S that maximizes $\mathrm{SRP}(\mathsf{S})$ and thus, in turn, μ_X. We shall illustrate this task in detail in Section 3 below. At the moment we note, for example, that when the prices X_L and X_H are independent, then Theorem 2.1 gives the equation

$$\mu_X = \mathrm{BEP} + p\mathbf{E}[\mathsf{S}(X_L)]\mathbf{E}[X_H] + (1-p)\mathbf{E}[\mathsf{S}(X_H)]\mathbf{E}[X_L]$$

$$-p\mathbf{E}[\mathsf{S}(X_L)X_L] - (1-p)\mathbf{E}[\mathsf{S}(X_H)X_H]. \qquad (2.1)$$

When X_L and X_H are dependent, and assuming for the sake of concreteness that $X_H = 2X_L$ as is the case in the two-envelope problem (more details in a moment), we have from Theorem 2.1 that

$$\mu_X = \mathrm{BEP}_2 + p\mathbf{E}[X_L\mathsf{S}(X_L)] - (1-p)\mathbf{E}[\mathsf{S}(2X_L)], \qquad (2.2)$$

where $\mathrm{BEP}_2 = (2-p)\mathbf{E}[X_L]$, which is called the benchmark base return by McDonnell and Abbott (2009).

More generally, when $X_H = \alpha X_L$ for a constant $\alpha > 1$, then

$$\mu_X = \mathrm{BEP}_\alpha + (\alpha-1)p\mathbf{E}[X_L\mathsf{S}(X_L)] - (\alpha-1)(1-p)\mathbf{E}[\mathsf{S}(\alpha X_L)] \qquad (2.3)$$

with the benchmark expected price $\mathrm{BEP}_\alpha = (\alpha-(\alpha-1)p)\mathbf{E}[X_L]$. Obviously, equation (2.3) implies (2.2) by setting $\alpha = 2$.

Now we recall the two-envelope problem. There are two individuals: a host and a player. The host randomly chooses an amount X_L of money and places it into one envelope and then places twice the amount into another envelope, that is,

$$X_H = 2X_L.$$

The two envelopes are indistinguishable. The player needs to decide whether to keep the received envelope or exchange it into another one. Once a decision has been made, the game is over and the host and the player keep the money that they find in their respective envelopes.

For an optimal strategy in this game, which has greatly influenced our present research, we refer to McDonnell and Abbott (2009), and McDonnell et al. (2011). It should be noted that there are many ways in which the two-envelope problem can be formulated, and the literature on the topic is vast. The assumptions of McDonnell and Abbott (2009) may not conform with all available versions of the two-envelope problem, but the framework of the noted paper has played a pivotal role in our current research.

3 The maximizing strategy

Under various scenarios, in this section we demonstrate how the maximizing strategy $\mathsf{S}_{\mathrm{MAX}}$ can be derived and how it looks like. The following corollary to Theorem 2.1 provides an explicit form of the strategy.

Corollary 3.1 Assume that X_L and X_H have densities f_L and f_H, respectively. Then the maximizing strategy function $\mathsf{S}_{\text{MAX}}(y)$ is the indicator $\mathbf{1}_A(y)$ of the set $A = \{x \in [0,\infty) : \mathsf{H}_{\text{MAX}}(x) > 0\}$, where

$$\mathsf{H}_{\text{MAX}}(x) = p\{\mathbf{E}[X_H \mid X_L = x] - x\} f_L(x) + (1-p)\{\mathbf{E}[X_L \mid X_H = x] - x\} f_H(x).$$

Proof. By the definition of $\text{SRP}(\mathsf{S})$ given in Theorem 2.1 and using the assumption that the random prices X_L and X_H have densities, we easily arrive at the equation

$$\text{SRP}(\mathsf{S}) = \int \mathsf{S}(x)\mathsf{H}_{\text{MAX}}(x)dx. \quad (3.1)$$

Since $\mathsf{S}(x)$ is always in the interval $[0,1]$, the maximizing strategy $\mathsf{S}_{\text{MAX}}(x)$ must be equal to 1 when $\mathsf{H}_{\text{MAX}}(x) > 0$ and 0 when $\mathsf{H}_{\text{MAX}}(x) \le 0$. In other words, $\mathsf{S}_{\text{MAX}}(x)$ must be the indicator function $\mathbf{1}_A(x)$ of the set A defined in the corollary. This finishes the proof of Corollary 3.1.

As we shall see in the following two subsections, the maximizing strategy $\mathsf{S}_{\text{MAX}}(y)$ is often a threshold type strategy and takes on the form $\mathsf{S}_b(y) = \mathbf{1}_{[0,b)}(y)$, where b is a `threshold' that maximizes $\text{SRP}(\mathsf{S}_b)$. This strategy means rejecting the first-come offer when it is smaller than b and accepting the first-come offer when it is equal or larger than b.

ACCEPTED MANUSCRIPT

3.1 When the prices X_L and X_H are independent

Throughout this subsection we assume that X_L and X_H are independent random variables. In this case, the maximizing strategy S_{MAX} is specified by Corollary 3.1 with the function

$$\mathsf{H}_{\mathrm{MAX}}(x) = p\{\mathbf{E}[X_H] - x\} f_L(x) + (1-p)\{\mathbf{E}[X_L] - x\} f_H(x). \quad (3.2)$$

We need to find those $x \geq 0$ for which $\mathsf{H}_{\mathrm{MAX}}(x) > 0$. The likelihood ratio stochastic dominance (e.g., Denuit et al. 2005; Furman and Zitikis 2008) plays a natural role here. Namely, we say that X_H is larger (or, rather, not smaller) than X_L in the likelihood ratio sense, written as $X_H \geq_{LR} X_L$, if

$$w(x) = \frac{f_H(x)}{f_L(x)} \text{ is a non-decreasing function of } x. \qquad (3.3)$$

To make the following considerations more transparent, we assume that X_L and X_H have same supports, say the interval (x_1, x_2) for some $0 \leq x_1 < x_2 \leq +\infty$. This means that the two densities $f_L(x)$ and $f_H(x)$ are (strictly) positive for all $x \in (x_1, x_2)$ and equal to 0 outside the interval (x_1, x_2).

ACCEPTED MANUSCRIPT

Note that assumption (3.3) implies that $\mu_H \geq \mu_L$, where $\mu_H = \mathbf{E}[X_H]$ and $\mu_L = \mathbf{E}[X_L]$. To prove this fact, we rewrite μ_H as follows:

$$\mu_H = \int x \frac{f_H(x)}{f_L(x)} f_L(x)dx = \mathbf{E}[X_L w(X_L)].$$

Consequently, verifying $\mu_H \geq \mu_L$ is equivalent to veifying $\mathbf{Cov}[X_L, w(X_L)] \geq 0$ because $\mathbf{E}[w(X_L)] = \int f_H(x)dx = 1$. But by Lehmann (1966) we know that the covariance $\mathbf{Cov}[X_L, w(X_L)]$ is non-negative because the function w is non-decreasing. For related results with possibly non-monotonic weight functions w, we refer to Egozcue et al. (2011), and references therein.

Theorem 3.1 Under the above assumptions on the densities f_L and f_H, and in particular assuming (3.3), we have that

$$\mathsf{S}_{\mathrm{MAX}}(y) = 1_{(x_1,b)}(y) \quad (3.4)$$

with the threshold $b = \sup\{x > \mu_L : v(x) > w(x)\}$, *where*

$$v(x) = \frac{p(\mu_H - x)}{(1-p)(x-\mu_L)}.$$

ACCEPTED MANUSCRIPT

In particular, when $\mu_L = \mu_H\ (\equiv \mu)$, *then* $\mathsf{S}_{\mathrm{MAX}}(y) = \mathbf{1}_{(x_1,\mu)}(y)$.

Proof. The assumptions on the densities f_L and f_H imply that the means μ_L and μ_H are in the interval (x_1, x_2). Moreover, we already know that $\mu_L \le \mu_H$. Keeping in mind that $f_L(x)$ and $f_H(x)$ are (strictly) positive for all $x \in (x_1, x_2)$ and equal to 0 outside the interval (x_1, x_2), we need to specify those $x \in (x_1, x_2)$ for which $\mathsf{H}_{\mathrm{MAX}}(x) > 0$. To this end, we consider the cases $\mu_L = \mu_H$ and $\mu_L < \mu_H$ separately; and there can only be these two cases.

When $\mu_L = \mu_H$, in which case we denote the two expectations by μ, we have $\mathsf{H}_{\mathrm{MAX}}(x) > 0$ if and only if $x < \mu$. Hence, the maximizing strategy is $\mathsf{S}_{\mathrm{MAX}}(y) = \mathbf{1}_{(x_1,\mu)}(y)$. This coincides with strategy (3.4) because when $\mu_L = \mu_H$, then $v(x) = -p/(1-p) < 0$ and thus the supremum $\sup\{x > \mu_L : v(x) > w(x)\}$ is calculated over the empty set of x values and thus, by the usually agreed definition, the supremum is set to the smallest value of x, which is $\mu \equiv \mu_L$.

When $\mu_L < \mu_H$, then $\mathsf{H}_{\mathrm{MAX}}(x) > 0$ for all $x \in (x_1, \mu_L]$ and $\mathsf{H}_{\mathrm{MAX}}(x) < 0$ for all $x \in [\mu_H, x_2)$. Hence, it remains to specify those $x \in (\mu_L, \mu_H)$ for which $\mathsf{H}_{\mathrm{MAX}}(x) > 0$. The latter bound is equivalent to $v(x) > w(x)$. The function $v(x)$ is decreasing on the interval (μ_L, μ_H): it starts with an infinite value at $x = \mu_L$ and ends with the value 0 at $x = \mu_H$. Since the function $w(x)$ is non-decreasing by assumption (3.3), we therefore must have a point $b \in (\mu_L, \mu_H)$ such that the bound $v(x) > w(x)$ holds for all $x \in (\mu_L, b)$, and the opposite bound $v(x) \le w(x)$ holds for all

$x \in [b, \mu_H)$. Consequently, $\mathsf{S}_{\text{MAX}}(y) = \mathbf{1}_{(x_1, b)}(y)$ with the threshold b defined in the formulation of Theorem 3.1. This concludes the proof of Theorem 3.1.

3.2 When $X_H = \alpha X_L$ for a constant $\alpha > 1$

Here we explore the case when the prices X_L and X_H are tied via the equation $X_H = \alpha X_L$ for some constant $\alpha > 1$. (The classical two-envelope problem corresponds to the case $\alpha = 2$.) Hence, in particular, $f_H(x) = (1/\alpha) f_L(x/\alpha)$ and so, by equation (3.1) and some little algebra, we obtain

$$\text{SRP}(\mathsf{S}) = (\alpha - 1)\int \mathsf{S}(x)x[pf_L(x) - (1-p)\frac{1}{\alpha^2} f_L(\frac{x}{\alpha})]dx. \qquad (3.5)$$

The maximizing strategy S_{MAX} is therefore given by Corollary 3.1 with the function

$$\mathsf{H}_{\text{MAX}}(x) = pf_L(x) - (1-p)\frac{1}{\alpha^2} f_L(\frac{x}{\alpha}), \qquad (3.6)$$

which in the case $\alpha = 2$ appears on p. 3316 of McDonnell and Abbott (2009) and also plays an important role throughout the paper of McDonnell et al. (2011).

We shall next illustrate the maximizing strategy $\mathsf{S}_{\mathrm{MAX}}$ in the case of three parametric distributions that have been noted by several authors dealing with real estate prices (e.g., Gastwirth 1976; Ohnishi et al. 2011).

3.2.1 Uniform distribution of prices

We start with the uniform on $[A,B]$ distribution, whose density is

$$f_L(x) = \frac{\mathbf{1}_{[A,B]}(x)}{B-A}$$

for some parameters $0 \le A < B < +\infty$.

Theorem 3.2 When X_L is uniform on $[A,B]$, then

$$\mathsf{S}_{\mathrm{MAX}}(y) = \begin{cases} \mathbf{1}_{[A,\alpha A)}(y) & \textit{when} \quad \alpha A \le B \quad \textit{and} \quad p \le 1/(1+\alpha^2), \\ \mathbf{1}_{[A,B]}(y) & \textit{otherwise.} \end{cases} \tag{3.7}$$

Proof. We need to specify those $x \in [A,B]$ for which

$$p\mathbf{1}_{[A,B]}(x) - \frac{1-p}{\alpha^2}\mathbf{1}_{[\alpha A,\alpha B]}(x) > 0.$$

This is equivalent to checking the bound

$$(\frac{1}{p}-1)\frac{1}{\alpha^2}\mathbf{1}_{[\alpha A,\alpha B]}(x) < \mathbf{1}_{[A,B]}(x).$$

By considering the cases $\alpha A > B$ and $\alpha A \leq B$ separately, with the latter case split into two subcases $p \leq 1/(1+\alpha^2)$ and $p > 1/(1+\alpha^2)$, we arrive at the strategy S_{MAX} given by equation (3.7). This completes the proof of Theorem 3.2.

We shall now decipher Theorem 3.2 in terms of the plain decision-making language. To begin with, the parameter (α, p) space is the strip $(1,\infty)\times[0,1]$. Define a subset of the strip:

$$\Delta = \left\{(\alpha, p) \in (1,\infty)\times[0,1] : \alpha \leq B/A,\ p \leq 1/(1+\alpha^2)\right\}.$$

Recall now that if the size y of a first-come offer is such that $\mathsf{S}_{\mathrm{MAX}}(y) = 1$, then the offer should be rejected, but if $\mathsf{S}_{\mathrm{MAX}}(y) = 0$, then it should be accepted. Hence, the decision rule:

•When $(\alpha, p) \notin \Delta$, then we always reject the first-come offer, irrespectively of its size y.

•When $(\alpha, p) \in \Delta$, then we reject the first-come offer of size y if $y < \alpha A$ and accept it if $y \geq \alpha A$.

To get a better feel for this result, here is a numerical example. Suppose that X_L follows the uniform on $[300, 320]$ (in thousands) distribution, and let α be an increase by 5%, that is,

ACCEPTED MANUSCRIPT

$\alpha = 1.05$. The condition $\alpha \le B/A$ is satisfied, and thus we only need to look into two cases: $p \le 1/(1+\alpha^2) = 0.4756$ and $p > 0.4756$. In the latter case, we always reject the first-come offer irrespectively of its size. In the former case, that is when the probability p of the first-come offer being from the lower distribution L is not large than 0.4756, we make the following decision: if $y < 315$, then we reject the first-come offer, but if $y \ge 315$, then we accept the offer.

3.2.2 Log-normal distribution of prices

Here we consider the random price X_L that follows the log-normal distribution, whose density is

$$f_L(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\{-\frac{(\log(x)-\mu)^2}{2\sigma^2}\} \mathbf{1}_{(0,\infty)}(x)$$

for some parameters $\mu \in (-\infty, \infty)$ and $\sigma > 0$.

Theorem 3.3 When X_L is log-normal, then

$$\mathsf{S}_{\mathrm{MAX}}(y) = \mathbf{1}_{(0,b)}(y)$$

with the threshold

$$b = \sqrt{\alpha}\exp\{\mu\}\left(\frac{p\alpha}{1-p}\right)^{\sigma^2/\log\alpha}. \ (3.8)$$

Proof. We check that $\mathsf{H}_{\mathrm{MAX}}(x)$ defined by equation (3.6) is positive if and only if

$$p\frac{1}{x\sigma\sqrt{2\pi}}\exp\left\{-\frac{(\log(x)-\mu)^2}{2\sigma^2}\right\} > \frac{1-p}{\alpha x\sigma\sqrt{2\pi}}\exp\left\{-\frac{(\log(x/\alpha)-\mu)^2}{2\sigma^2}\right\}.$$

Canceling out some terms and taking the logarithms of both sides, the above inequality becomes

$$-\frac{(\log(x)-\mu)^2}{2\sigma^2} > \log\left(\frac{1-p}{p\alpha}\right) - \frac{(\log(x/\alpha)-\mu)^2}{2\sigma^2},$$

which is equivalent to the following one:

$$x < \exp\left\{\mu - \frac{\sigma^2}{\log\alpha}\log\left(\frac{1-p}{p\alpha}\right) + \frac{\log\alpha}{2}\right\}$$

$$= \sqrt{\alpha}\exp\{\mu\}\left(\frac{p\alpha}{1-p}\right)^{\sigma^2/\log\alpha}.$$

The right-hand side is the threshold b, thus finishing the proof of Theorem 3.3.

3.2.3 Pareto distribution of prices

Our final example deals with the price X_L that follows the Pareto distribution, whose density is

$$f_L(x) = \frac{\theta}{x_0}(\frac{x_0}{x})^{\theta+1}\mathbf{1}_{[x_0,+\infty)}(x)$$

for some parameters $x_0 > 0$ and $\theta > 1$. Note that the restriction $\theta > 1$ is necessary for the finiteness of the first moment of X_L, which we need. The preference of the Pareto distribution over the lognormal distribution when modeling house prices has been noted by Ohnishi et al. (2011).

Theorem 3.4 When X_L is Pareto, then

$$\mathsf{S}_{\mathrm{MAX}}(y) = \begin{cases} \mathbf{1}_{[x_0,\alpha x_0)}(y) & \textit{when} \quad p \le 1/(1+\alpha^{1-\theta}), \\ \mathbf{1}_{[x_0,+\infty)}(y) & \textit{otherwise.} \end{cases}$$

Proof. We need to know when $\mathsf{H}_{\mathrm{MAX}}(x)$ defined by equation (3.6) is positive. This is equivalent to checking the inequality

$$\mathbf{1}_{[x_0,+\infty)}(x) > \frac{1-p}{p}\alpha^{\theta-1}\mathbf{1}_{[\alpha x_0,+\infty)}(x).$$

When $x \in [x_0, \alpha x_0)$, then this inequality always holds, but when $x \in [\alpha x_0, \infty)$, then it holds if and only if $p > 1/(1+\alpha^{1-\theta})$. This completes the proof of Theorem 3.4.

4 Concluding notes

The real estate business is a fascinating laboratory for testing theories and techniques of decision theory, economics, probability, psychology, sociology, and other research areas. It also touches upon several problems, or puzzles, that have fascinated amateur and professional scientists. In this paper, whose main contribution is an optimal real-estate seller's strategy in the motivating problem, we have noted a connection between the motivating problem and the well-known two-envelope problem, in the form of McDonnell and Abbott (2009), and McDonnell et al. (2011).

Another closely related problem to developing strategies in the real estate business is the secretary problem, as noted and utilized by Mazalov and Saario (2002), who derived an optimal threshold-type strategy for setting selling prices under the assumption of the sequential arrival of buyers. Mazalov and Saario (2002) assume (for the sake of mathematical simplicity) that the prices are uniformly distributed but their ideas can be extended to other distributions as well.

One can find many fascinating connections between the real estate business and other problems or puzzles of decision theory and related areas, but this has not been the main goal of the present paper.

ACCEPTED MANUSCRIPT

Acknowledgments

Sincere thanks are due to two anonymous referees for their interest, suggestions, and criticism that have guided us when revising the paper. We are grateful to Andrew Crook, Broker of Record, and Dustin Pritchard, Sales Manager, as well as the dedicated staff of the Pritchard & Crook Real Estate Brokerage Inc., London, Ontario, for enhancing our understanding of the intricacies of the real estate business. The efforts of Jacob Petronis, Sales Representative, are very much valued. This research has been partially supported by the Agencia Nacional de Investigación e Innovación (ANII) of Uruguay, and the Natural Sciences and Engineering Research Council (NSERC) of Canada.

References

Albright, S.C. (1977). A Bayesian approach to a generalized house selling problem. *Management Science*, 24, 432–440.

Anglin, P.M. (2004). How long does it take to buy one house and sell another? *Journal of Housing Economics*, 13, 87–100.

Anglin, P.M., Rutherford, R. and Springer, T.M. (2003). The trade-off between the selling price of residential properties and time-on-the-market: the impact of price setting. *Journal of Real Estate Finance and Economics*, 26 95–111.

Arnold, M.A. (1999). Search, bargaining and optimal asking prices. *Real Estate Economics*, 27 453–481.

Biswas, T. and McHardy, J. (2007). Asking price and price discounts: the strategy of selling an asset under price uncertainty. *Theory and Decision*, 62 281–301.

Brint, A. (2009). Predicting a house's selling price through inflating its previous selling price. *Journal of the Operational Research Society*, 60, 339–347.

Cheng, P., Lin, L. and Liu, Y. (2008). A model of time-on-market and real estate price under sequential search with recall. *Real Estate Economics*, 36, 813–843.

DeGroot, M.H. (1970). *Optimal Statistical Decisions*. McGraw-Hill, New York.

Deng, Y., Gabriel, S.A., Nishimura, K.G. and Zheng, D. (2013). Optimal pricing strategy in the case of price dispersion: new evidence from the Tokyo housing market. *Real Estate Economics*, 41. (To appear.)

Denuit, M., Dhaene, J., Goovaerts, M. and Kaas, R. (2005). *Actuarial Theory for Dependent Risks: Measures, Orders and Models*. Wiley, Chichester.

Egozcue, M., Fuentes García, L., Wong, W.K. and Zitikis, R. (2011). The covariance sign of transformed random variables with applications to economics and finance. *IMA Journal of Management Mathematics*, 22, 291–300.

Furman, E. and Zitikis, R. (2008). Weighted premium calculation principles. *Insurance: Mathematics and Economics*, 42, 459–465.

ACCEPTED MANUSCRIPT

Gastwirth, J.L. (1976). On probabilistic models of consumer search for information. *Quarterly Journal of Economics*, 90, 38–50.

Glower, M., Haurin, D.R. and Hendershott, P.H. (1998). Selling time and selling price: the influence of investor motivation. *Real Estate Economics*, 26, 719–740.

Horowitz, J.L. (1992). The role of the list price in housing markets: theory and an econometric model. *Journal of Applied Econometrics*, 7, 115–129.

Jares, T.E., Larsen, J.E. and Zorn, T.S. (2000). An optimal incentive system for real estate agents. *Journal of Real Estate Research*, 20, 49–59.

Larsen, J.E. and Park, W.J. (1989). Non-uniform percentage brokerage commissions and real estate market performance. *Journal of the American Real Estate and Urban Economics Association*, 17, 422–438.

Lehmann, E.L. (1966). Some concepts of dependence. *Annals of Mathematical Statistics*, 37, 1137–1153.

Lippman, S.A. and McCall, J.J. (1976). The economics of job search: a survey. *Economic Inquiry*, 14, 155–189.

Mazalov, V.V. and Saario, V. (2002). The house-selling problem with reward rate criterion. *Journal of Applied Probability*, 39, 644–649.

McDonnell, M.D. and Abbott, D. (2009). Randomized switching in the two-envelope problem. *Proceedings of the Royal Society A*, 465, 3309–3322.

ACCEPTED MANUSCRIPT

McDonnell, M.D., Grant, A.J., Land, I., Vellambi, B.N., Abbott, D., and Lever, K. (2011). Gain from the two-envelope problem via information asymmetry: on the suboptimality of randomized switching. *Proceedings of the Royal Society A*, 467, 2825–2851.

Miceli, T.J. (1989). The optimal duration of real estate listing contracts. *Journal of the American Real Estate and Urban Economics Association*, 17, 267–277.

Nelson, P. (1970). Information and consumer behavior. *Journal of Political Economy*, 78, 311–329.

Ohnishi, T., Mizuno, T., Shimizu, C. and Watanabe, T. (2011). The evolution of house price distribution. *RIETI Discussion Paper*, Series 11-E-019, 20 pages.

Read, C. (1988). Price strategies for idiosyncratic goods – the case of housing. *Journal of the American Real Estate and Urban Economics Association*, 16, 379–395.

Riley, J. and Zeckhauser, R. (1983). Optimal selling strategies: when to haggle, when to hold firm. *Quarterly Journal of Economics*, 98, 267–289.

Rosenfield, D.B., Shapiro, R.D. and Butler, D.A. (1983). Optimal strategies for selling an asset. *Management Science*, 29, 1051–1061.

Salant, S.W. (1991). For sale by owner: when to use a broker and how to price the house. *Journal of Real Estate Finance and Economics*, 4, 157–173.

Sirmans, C.F., Turnbull, G.K. and Dombrow, J. (1995). Quick house sales: investor mistake or luck? *Journal of Housing Economics*, 4, 230–243.

ACCEPTED MANUSCRIPT

Stigler, G.J. (1961). The economics of information. *Journal of Political Economy*, 69, 213–225.

Yavas, A. and Yang, S. (1995). The strategic role of listing price in marketing real estate: theory and evidence. *Real Estate Economics*, 23, 347–368.

Zorn, T.S. and Larsen, J.E. (1986). The incentive effects of flat-fee and percentage commissions for real estate brokers. *Journal of the American Real Estate and Urban Economics Association*, 14, 24–47.

A Appendix: Proof of Theorem 2.1

The proof is based on the rule of total probability and conditioning arguments, and we start out with the equation $\mu_X = \mathbf{E}[\mathbf{E}[X \mid X_L, X_H]]$. To calculate the conditional expectation $\mathbf{E}[X \mid X_L = x, X_H = y]$, note that under the condition $X_L = x, X_H = y$, the ultimate selling price X can only be either x or y. Hence,

$$\mathbf{E}[X \mid X_L = x, X_H = y] = x\mathbf{P}[X = x \mid X_L = x, X_H = y] + y\mathbf{P}[X = y \mid X_L = x, X_H = y].$$

(A.1)

We next calculate the two probabilities on the right-hand side of equation (A.1), and start with the first probability. Using the random variable O_1 of `first offer,' we write

$$\mathbf{P}[X = x \mid X_L = x, X_H = y]$$

$$= \mathbf{P}[X = x \mid X_L = x, X_H = y, O_1 = L]\mathbf{P}[O_1 = L \mid X_L = x, X_H = y]$$

$$+\mathbf{P}[X = x \mid X_L = x, X_H = y, O_1 = H]\mathbf{P}[O_1 = H \mid X_L = x, X_H = y]. \qquad \text{(A.2)}$$

Next we employ the random variable R_1 of `rejecting the first offer' and have the equation

$$\mathbf{P}[X = x \mid X_L = x, X_H = y, O_1 = L]$$

$$= \mathbf{P}[X = x \mid X_L = x, X_H = y, O_1 = L, R_1 = Y]\mathbf{P}[R_1 = Y \mid X_L = x, X_H = y, O_1 = L]$$

$$+ \mathbf{P}[X = x \mid X_L = x, X_H = y, O_1 = L, R_1 = N]\mathbf{P}[R_1 = N \mid X_L = x, X_H = y, O_1 = L].$$

Observe that the first probability on the right-hand side of the above equation is equal to 0 whereas the last probability is equal to 1. Hence,

$$\mathbf{P}[X = x \mid X_L = x, X_H = y, O_1 = L] = \mathbf{P}[R_1 = N \mid X_L = x, X_H = y, O_1 = L]. \qquad \text{(A.3)}$$

Analogously,

$$\mathbf{P}[X = x \mid X_L = x, X_H = y, O_1 = H] = \mathbf{P}[R_1 = Y \mid X_L = x, X_H = y, O_1 = H]. \qquad \text{(A.4)}$$

Using equations (A.3) and (A.4) on the right-hand side of equation (A.2), we have that

ACCEPTED MANUSCRIPT

$$\mathbf{P}[X = x \mid X_L = x, X_H = y]$$

$$= \mathbf{P}[R_1 = N \mid X_L = x, X_H = y, O_1 = L]\mathbf{P}[O_1 = L \mid X_L = x, X_H = y]$$

$$+\mathbf{P}[R_1 = Y \mid X_L = x, X_H = y, O_1 = H]\mathbf{P}[O_1 = H \mid X_L = x, X_H = y]. \qquad \text{(A.5)}$$

Analogously, or simply by making the two notational changes $x \leftrightarrow y$ and $L \leftrightarrow H$ in the above equation, we obtain

$$\mathbf{P}[X = y \mid X_L = x, X_H = y]$$

$$= \mathbf{P}[R_1 = N \mid X_L = x, X_H = y, O_1 = H]\mathbf{P}[O_1 = H \mid X_L = x, X_H = y]$$

$$+\mathbf{P}[R_1 = Y \mid X_L = x, X_H = y, O_1 = L]\mathbf{P}[O_1 = L \mid X_L = x, X_H = y]. \qquad \text{(A.6)}$$

Using equations (A.5) and (A.6) on the right-hand side of equation (A.1), we have

$$\mathbf{E}[X \mid X_L = x, X_H = y]$$

$$= x\mathbf{P}[R_1 = N \mid X_L = x, X_H = y, O_1 = L]\mathbf{P}[O_1 = L \mid X_L = x, X_H = y]$$

$$+ x\mathbf{P}[R_1 = Y \mid X_L = x, X_H = y, O_1 = H]\mathbf{P}[O_1 = H \mid X_L = x, X_H = y]$$

$$+\, y\mathbf{P}[R_1 = N \mid X_L = x, X_H = y, O_1 = H]\mathbf{P}[O_1 = H \mid X_L = x, X_H = y]$$

$$+\, y\mathbf{P}[R_1 = Y \mid X_L = x, X_H = y, O_1 = L]\mathbf{P}[O_1 = L \mid X_L = x, X_H = y]. \quad \text{(A.7)}$$

By Assumption 2.1, the probability $\mathbf{P}[O_1 = L \mid X_L = x, X_H = y]$ does not depend on x and y, and is equal to p. In view of this, equation (A.7) becomes

$$\mathbf{E}[X \mid X_L = x, X_H = y] = px\mathbf{P}[R_1 = N \mid X_L = x, X_H = y, O_1 = L]$$

$$+\,(1-p)x\mathbf{P}[R_1 = Y \mid X_L = x, X_H = y, O_1 = H]$$

$$+\,(1-p)y\mathbf{P}[R_1 = N \mid X_L = x, X_H = y, O_1 = H]$$

$$+\, py\mathbf{P}[R_1 = Y \mid X_L = x, X_H = y, O_1 = L]. \quad \text{(A.8)}$$

By Assumption 2.1, the probability $\mathbf{P}[R_1 = Y \mid X_L = x, X_H = y, O_1 = H]$ is equal to $\mathsf{S}(y)$, and thus the probability $\mathbf{P}[R_1 = N \mid X_L = x, X_H = y, O_1 = L]$ is equal to $1-\mathsf{S}(x)$. Analogous formulas for the other two probabilities on the right-hand side of equation (A.8) are valid. Hence, from equation (A.8) we have that $\mathbf{E}[X \mid X_L = x, X_H = y]$ is equal to

$$x\big(p(1-\mathsf{S}(x)) + (1-p)(y)\big) + y\big(p\mathsf{S}(x) + (1-p)(1-\mathsf{S}(y))\big),$$

ACCEPTED MANUSCRIPT

and so we have arrived at the equation

$$\mathbf{E}[X \mid X_L, X_H] = X_L\left((p(1-\mathsf{S}(X_L)) + (1-p)\mathsf{S}(X_H)\right)$$

$$+X_H\left((p\mathsf{S}(X_L) + (1-p)(1-\mathsf{S}(X_H))\right). \qquad \text{(A.9)}$$

Using equation (A.9), we obtain from $\mu_X = \mathbf{E}[\mathbf{E}[X \mid X_L, X_H]]$ that $\mu_X = \mathrm{BEP} + \mathrm{SRP}(\mathsf{S})$ with the strategy risk parameter

$$\mathrm{SRP}(\mathsf{S}) = \mathbf{E}\left[(X_H - X_L)(p\mathsf{S}(X_L) - (1-p)\mathsf{S}(X_H))\right]$$

$$= p\mathbf{E}[(X_H - X_L)\mathsf{S}(X_L)] - (1-p)\mathbf{E}[(X_H - X_L)\mathsf{S}(X_H)]. \qquad \text{(A.10)}$$

We next apply the equations $\mathbf{E}[X_H\mathsf{S}(X_L)] = \mathbf{E}[\mathbf{E}[X_H \mid X_L]\mathsf{S}(X_L)]$ and $\mathbf{E}[X_L\mathsf{S}(X_H)] = \mathbf{E}[\mathbf{E}[X_L \mid X_H]\mathsf{S}(X_H)]$ on the right-hand side of equation (A.10) and arrive at the expression of $\mathrm{SRP}(\mathsf{S})$ given in the formulation of the theorem. This completes the proof of Theorem 2.1.