Accuracy assessment of a high-order moving least squares finite volume method for compressible flows

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Abstract
This paper proposes an investigation of some important properties of a high-order finite-volume moving least-squares based method (FV-MLS) for the solution of two-dimensional Euler and Navier-Stokes equations on unstructured grids. A particular attention is paid to the computation of derivatives of shape functions by means of diffuse or full discretization. Furthermore, we introduce the semi-diffuse approach, which is a compromise in terms of accuracy and computational cost. In addition, we investigate the influence of the curvature of wall boundary conditions on the proposed numerical scheme. As expected, we found an improvement when the walls’ curvature is taken into account. However, and differently as in discontinuous Galerkin schemes, the use of a straight representation of the wall normals does not induce a substantial loss of accuracy. Numerical simulations show the accuracy and the robustness of the numerical approach for both inviscid and viscous flows.

Keywords: Compressible flows, unstructured grid, high-order finite volume scheme, moving least-squares

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1. Introduction

Achieving high order of accuracy on unstructured grids remains a great challenge to effectively capture complex flow features in computational fluid dynamics and computational aeroacoustics. During the last decade, high-order continuous finite element methods (FEM), discontinuous Galerkin methods (DGM) and spectral volume methods (SVM) have gained popularity for the numerical simulation of compressible Euler and Navier-Stokes equations [40]. On the contrary, the field of research for higher-order unstructured finite-volume schemes is less active, although second-order finite-volume schemes are routinely used for engineering problems. The key issue in the development of high-order unstructured finite-volume schemes is the implementation of efficient reconstruction procedures of unknown variables at the interface of the control volumes.

The most popular approaches employed to construct third- and fourth-order finite-volume scheme for hyperbolic conservation laws on unstructured grids are $k$-exact least-squares reconstruction [2, 14, 8, 30, 39, 27, 32] and ENO [1, 14, 33] or WENO [12, 17, 36, 9] reconstructions. The former stipulates that any solution which is expressed as a polynomial function of degree $k$ is reconstructed exactly at the cell interface. By its design, $k$-exact least-squares reconstruction properly ensures conservation of the mean, which requires that the average of the reconstructed solution over the control volume is equal to the average of the original function [32]. The method can be applied to curved geometries without deteriorating the high-order accuracy by means of careful treatment of flux integrations and control volume moments [31]. An efficient implicit matrix-free Newton-GMRES methods for fourth-order $k$-exact discretization of two dimensional steady Euler equations is developed in [27]. On the other hand, FV-WENO schemes combine the efficiency of their finite-difference counterparts with the flexibility of irregular grids. Third- and fourth-order WENO schemes, based on linear and quadratic polynomials for triangular meshes, are successfully applied to various unsteady inviscid flows in [12, 17]. Dumbser et al. [9] constructed a fourth-order quadrature free WENO scheme for nonlinear hyperbolic systems on unstructured tetrahedral meshes for straight geometries.

Recently, Cueto-Felgueroso et al. [6, 7] proposed to evaluate the successive derivatives involved in the high-order reconstruction step by means of moving least squares (MLS) approximations [22]. The interpolation structure is obtained from a weighted least-squares fitting of the solution based on
a given kernel function. As a consequence, the reconstructed solution is not a polynomial. Furthermore, the resulting FV-MLS method allows a direct reconstruction of the high-order viscous fluxes and the mean conservation is automatically satisfied for steady-state computations. A comparative study with a discontinuous Galerkin scheme for the solution of the Navier-Stokes equations on quadrilateral meshes can be found in [29].

This paper proposes an investigation of some important properties of the FV-MLS method for the solution of two-dimensional Euler and Navier-Stokes equations on unstructured triangular grids. First, we focus on the computation of derivatives of the shape function. In previous studies [6, 7, 29], the derivatives of MLS shape functions were approximated using diffuse derivatives [18]. Even though using this approach, the convergence rate of the numerical scheme is conserved, no work has been made about the influence on the accuracy of the method. Here, we investigate the influence of the computation of the derivatives using the diffuse or the full discretization. In addition, we introduce the semi-diffuse approach, which is a compromise in terms of accuracy and computational cost. On the other hand, the influence of curved boundaries on the accuracy of high order methods has been pointed out by many authors [21, 9, 24, 27, 13]. In particular, it has been shown that Discontinuous Galerkin method is particularly sensitive, and straight representation of curved boundaries introduces large errors in the solution [3, 21]. Here, we investigate the influence of the curvature of wall boundary conditions on the formal order of accuracy in the context of moving least squares based finite volume methods. Numerical simulations demonstrate the accuracy and the robustness of the high-order FV-MLS method for both inviscid and viscous flows.

The outline of the paper is as follows. First, the finite-volume formulation based on moving least-squares reconstruction is described in section 2. Then, the accuracy assessment for smooth inviscid flow around a circular cylinder is shown in section 3. Finally, numerical experiments for separated viscous flows are presented for various configurations in section 4 and section 5.

2. Finite-volume formulation of the governing equations

The integral form of two-dimensional compressible Navier-Stokes equations over a bounded domain of interest reads

$$\frac{\partial}{\partial t} \int_{\Omega} Q \, dV + \int_{\partial \Omega} (F(Q) - G(Q)) \cdot \hat{n} \, ds = 0$$

(1)
where $\Omega$ represents the control volume and $\hat{n} = (n_x, n_y)^T$ denotes the outer unit normal vector to the boundary $\partial\Omega$. The vector of conservative variables $Q$ and the inviscid flux vector $F = (F_x, F_y)^T$ are given by

$$Q = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \quad F_x(Q) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E + p) \end{pmatrix}, \quad F_y(Q) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E + p) \end{pmatrix},$$

(2)

where $\rho$ is the density, $u, v$ are $x$-wise and $y$-wise components of the velocity vector, and $p, E$ denote the pressure and the total energy respectively. Cartesian components of the viscous fluxes are given by

$$G_x(Q) = \mu \begin{pmatrix} 2u_x - \frac{2}{3}(u_x + v_y) \\ v_x + u_y \\ u(2u_x - \frac{2}{3}(u_x + v_y)) + v(v_x + u_y) + C_p \frac{T}{T_x} \\ 0 \end{pmatrix},$$

$$G_y(Q) = \mu \begin{pmatrix} 0 \\ v_x + u_y \\ u(v_x + u_y) + v(2v_y - \frac{2}{3}(u_x + v_y)) + C_p \frac{T}{T_y} \end{pmatrix},$$

(3)

where $\mu$ denotes the dynamic molecular viscosity, $T$ is the temperature, $Pr$ is the Prandtl number and $C_p$ denotes the specific heat at constant temperature. The Sutherland’s law is employed to compute the dynamic viscosity

$$\mu = \mu_\infty \frac{T + S_0}{T_\infty + S_0} \left( \frac{T}{T_\infty} \right)^{1.5},$$

(4)

where $\mu_\infty$ and $T_\infty$ are the free-stream viscosity and temperature and $S_0 = 110.4 K$. The equation of state for an ideal gas is used to close the system of equations

$$p = (\gamma - 1) \left[ E - \rho \left( \frac{u^2 + v^2}{2} \right) \right],$$

(5)

where the ratio of specific heats is $\gamma = 1.4$. The governing equations are discretized using a cell-centered finite-volume formulation on an arbitrary unstructured grid. The problem domain $\Omega$ is divided in non-overlapping
triangular cells $\Omega_i$ whose number of face is denoted by $n_f$. For each control volume with faces $\Gamma_\ell$ ($\ell = 1, \ldots, 3$), Eq. (1) can be written as

$$\frac{\partial \bar{Q}_i}{\partial t} + \sum_{\ell=1}^{n_f} \int_{\Gamma_\ell} \left[ \mathbf{F}(\mathbf{Q}(\vec{x}, t)) - \mathbf{G}(\mathbf{Q}(\vec{x}, t)) \right] \cdot \hat{n} \, ds = 0$$

(6)

where $Q_i$ denotes the cell-averaged value of the solution

$$\bar{Q}_i = \frac{1}{|\Omega_i|} \int_{\Omega_i} \mathbf{Q}(\vec{x}, t) \, dv$$

(7)

The high-order flux integrals involved in equation (6) must be computed in such way that the accuracy of flux integration should be at least equal than the accuracy of the reconstruction of the solution [32]. In this work, we employ Gauss-Legendre quadratures. Therefore, a quadrature rule with $n_q$ points integrate exactly a polynomial of degree $2n_q - 1$. The spatially discretized form for the semi-discrete finite volume equations (6) can be written in short form as

$$\frac{d\bar{Q}_i}{dt} = -\frac{1}{|\Omega_i|} \sum_{\ell=1}^{n_f} \sum_{q=1}^{n_q} |\Gamma_\ell| w_q \left[ \mathbf{F}(\mathbf{Q}(\vec{x}_q, t)) - \mathbf{G}(\mathbf{Q}(\vec{x}_q, t)) \right] \cdot \hat{n} \equiv R_i(\bar{Q})$$

(8)

where $\vec{x}_q$ denotes the quadrature point with quadrature weight $w_q$ and $R_i(\bar{Q})$ represents the residual for the $i$-th cell. A single quadrature point situated at mid-side of the face with unit weight is considered for linear reconstruction. Two equally-weighted quadrature points per face are employed for the case of quadratic reconstruction. The normal fluxes across each element face in (8) can be evaluated using any suitable numerical fluxes $\mathbf{F}^{num}$ based on the right and left Riemann states of cell face

$$\mathbf{F}(\mathbf{Q}).\hat{n} \approx \mathbf{F}^{num}(\mathbf{Q}_L, \mathbf{Q}_R, \hat{n})$$

(9)

where subscripts $L$ and $R$ indicate the states of the flow properties at the right- and left-sides of the cell face. In this work, we employ the Roe’s flux difference scheme [35]. In order to obtain a third-order accurate scheme in space, the flow variable at both sides of a face is computed using the following quadratic component-wise reconstruction of any primitive variables $q$ of cell $\Omega_i$:

$$q(\vec{x}) = \bar{q}_i + \nabla \bar{q}_i \cdot \mathbf{r} + \frac{1}{2} \mathbf{r}^T \mathbf{H}_i \mathbf{r}$$

(10)
where \( r = (x - x_i) \) and \( H_i \) is the hessian matrix at cell centroid \( x_i \).

The semi-discrete system (8) is marched in time using an explicit three-stage third-order TVD Runge-Kutta scheme [37]. Since we only are interested in this study with steady state flows, computations are accelerated using local time stepping.

3. Solution reconstruction using Moving least-squares approximation

In this work, we seek to compute the derivatives involved in the Taylor’s reconstruction of (10) and in the viscous fluxes (3) by means of moving least-squares approximations. First, the general formulation of the moving least-squares method is presented. Then we discuss in details the construction of the MLS stencils in subsection 3.2. The procedure employed to perform a high-order flux integration on curved boundary is described in subsection 3.3.

3.1. Overview of the Moving Least-Squares Reproducing Kernel method

Let \( \Omega_x \) a compact support with \( n_x \) grid nodes, the MLS approximation \( \hat{q}(x) \) of \( q(x) \) reads [7]

\[
\hat{q}(x) = p^T(x)\alpha(z)|_{z=x} = N^T(x)q_{\Omega_x}
\]

(11)

where \( p^T(x) \) represents an \( m \)-dimensional functional basis, \( N^T(x) \) is the vector of MLS shape functions, \( q_{\Omega_x} = [q(x_1) \ q(x_2) \ ... \ q(x_{n_x})]^T \) contains the values of \( q(x) \) at each node of \( \Omega_x \) and the set of of parameters \( \alpha(z)|_{z=x} \), which result from the weighted least-squares fitting of \( q(x) \), are obtained through the minimization of the following error functional

\[
J(\alpha(z)|_{z=x}) = \int_{y \in \Omega_x} W(z-y, h)|_{z=x} (q(y) - p^T(y)\alpha(z)|_{z=x})^2 d\Omega_y
\]

(12)

In this work, a cubic spline kernel \( W(z-y, h) \) with circular support centered at \( z - x \) has been employed

\[
W(z-y, h) = \begin{cases} 
1 - \frac{3}{2}s^2 + \frac{3}{4}s^3 & s \leq 1 \\
\frac{1}{4}(2-s)^3 & 1 < s \leq 2 \\
0 & s > 2 
\end{cases}
\]

(13)
where \( s = ||\mathbf{x} - \mathbf{y}||/h \). The smoothing length parameter \( h \) is taken as \( h = \kappa d_{\text{max}} \) where \( d_{\text{max}} \) denotes the maximum distance between any node \( \mathbf{x} \) of \( \Omega_x \) and the reference node \( \mathbf{y} \) and \( \kappa \) is a shape parameter. In this study, we use \( \kappa = 0.65 \) according to the Fourier analysis of the FV-MLS conducted in [29] in the case of the one dimensional linear convection equation. From a practical point of view, the use of \( p \)th order complete polynomial basis as the functional basis \( \mathbf{p}^T(\mathbf{x}) \) results in a formal order of accuracy of \( p \) and \( p - 1 \) for the first and second derivatives respectively [29]. For two dimensional numerical simulations, the following \( p = 2 \) polynomial basis was used to computed the derivatives involved in the quadratic reconstruction defined by (10)

\[
\mathbf{p}(\mathbf{x}) = (1 \ x \ y \ xy \ x^2 \ y^2)^T
\]

Evaluating the integral in (12) using the nodes inside the support as quadrature points yields the following expression of \( \mathbf{N}^T(\mathbf{x}) \)

\[
\mathbf{N}^T(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) M^{-1}(\mathbf{x}) \mathbf{p}_{\Omega_x} W(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) C(\mathbf{x})
\]

where the moment matrix \( M(\mathbf{x}) \), the matrix of weight function \( W(\mathbf{x}) \) and \( \mathbf{p}_{\Omega_x} \) as given by

\[
\begin{align*}
M(\mathbf{x}) & = \mathbf{p}_{\Omega_x} W(\mathbf{x}) \mathbf{p}^T_{\Omega_x} \quad (16) \\
\mathbf{p}_{\Omega_x} & = [\mathbf{p}(\mathbf{x}_1) \mathbf{p}(\mathbf{x}_2) ... \mathbf{p}(\mathbf{x}_{n_x})]^T \\
W(\mathbf{x}) & = \text{diag} \{ W_i(\mathbf{x} - \mathbf{x}_i) \} \quad i = 1, ..., n_x
\end{align*}
\]

Note that scaled and locally defined monomials are used in the polynomial basis in order to enforce the conditioning of the moment matrix [7]. Deeper investigations about the FV-MLS methodology are presented in [6, 7, 29]. Using the definition of the MLS approximation given by (11), the derivatives of \( q(\mathbf{x}) \) are evaluated at node \( \mathbf{x}_i \) using the following expressions

\[
\begin{align*}
\frac{\partial q(\mathbf{x})}{\partial \mathbf{x}} \bigg|_{\mathbf{x}=\mathbf{x}_i} & = q_{\Omega_x} \frac{\partial N^T(\mathbf{x})}{\partial \mathbf{x}} \bigg|_{\mathbf{x}=\mathbf{x}_i}, \\
\frac{\partial^2 q(\mathbf{x})}{\partial \mathbf{x}^2} \bigg|_{\mathbf{x}=\mathbf{x}_i} & = q_{\Omega_x} \frac{\partial^2 N^T(\mathbf{x})}{\partial \mathbf{x}^2} \bigg|_{\mathbf{x}=\mathbf{x}_i}
\end{align*}
\]

where \( \partial N^T(\mathbf{x})/\partial \mathbf{x} \) can be derived as

\[
\frac{\partial N^T(\mathbf{x})}{\partial \mathbf{x}} = \frac{\partial \mathbf{p}^T(\mathbf{x})}{\partial \mathbf{x}} C(\mathbf{x}) + \mathbf{p}^T(\mathbf{x}) \frac{\partial C(\mathbf{x})}{\partial \mathbf{x}}
\]

After some algebra, the derivative of \( C(\mathbf{x}) \) can be expressed as

\[
\frac{\partial C(\mathbf{x})}{\partial \mathbf{x}} = C(\mathbf{x}) W^{-1}(\mathbf{x}) \frac{\partial W(\mathbf{x})}{\partial \mathbf{x}} (I - \mathbf{p}^T(\mathbf{x}) C(\mathbf{x}))
\]

Examination of (20) and (21) shows that the additional effort relative to
the computation of the flow gradient remains moderate because it involves
only the computation of the derivatives of the polynomial basis and of the
kernel functions. Conversely, the computations of the second derivatives of
\( C(x) \), which are required for the high-order reconstruction, lead to large
coding effort and computational costs as shown by the expressions of the full
derivatives of the MLS shape functions for quadratic reconstruction which
are given in Appendix. As a consequence, these terms are often neglected,
resulting in the construction of diffuse derivatives \([18, 6, 7]\). However, since
the first derivatives of \( C(x) \) are already known, we propose, in this work, to
account for these terms in order to construct the semi-diffuse approximation
of the second derivatives of \( N_T(x) \). A critical discussion about the influence
of the diffuse, semi-diffuse and full approaches on the formal order of accuracy
of the third-order scheme is given in section 4.2.

3.2. Building the reconstruction stencil

The computation of the shape function and its derivatives at the cell
centroids requires the knowledge of surrounding nodes forming a compact
support (namely the reconstruction stencil or MLS stencil). The construction
of the stencil is of crucial importance in the context of high-order finite
volume methods on unstructured grids \([7, 32, 4]\). It must be constructed
to avoid an ill-conditioned moment matrix \( M(x) \) and its number of control
volumes must be a compromise between affordable computational cost and
solution accuracy. Let \( x_i \) the centroid of the active cell \( C_i \), then \( j \) are the
centroids of the \( n_x \) surrounding cells. The local stencil is constructed by
successfully adding neighbor elements sharing a face with elements belonging
to the previous layer (Fig. 1 on the left). The minimum size of the stencil is
dictated by \((p+1)(p+2)/2\) grid nodes for a \( p \)th order polynomial basis \([7, 32]\).
However, the size of the supporting nodes may be increased in practice in
order to enforce the robustness of the numerical method. In the present
work, the MLS stencils comprise 8 and 15 control volumes for second-order
\( (p = 1) \) and third order \( (p = 2) \) accuracy, respectively.

Once the size of the stencil is defined, the step after concerns the dis-
tribution of the particles (centroids) around the active cell. Regardless to
the meshing strategy (tri, quad or hybrid), if possible, priority is given to
construct stencils as compact as possible by the use of layers of cells around
the active cell. But in practice the fulfillment of the compactness condition
requires a high number of points in the stencil, which induces sometime some stability problems and also unnecessary dissipation. To overcome this inconvenient, last particles which can not form a compact layer are placed such as satisfying a barycentric equilibrium (see fig. 1). Note that there are other possible choices, maybe more rigorous and more accurate. One possibility is, for example, to define a distribution criterion based on singularity-avoiding [4]. Other possible choices may be based on the best momentum matrix conditioning or the minimum of dissipation/dispersion stencil inducing properties. For highly stretched grids, priority on searching adjacent cells is given to the geometrical proximity and not to the ordering proximity.

![Figure 1: Typical 3rd order MLS stencils : a) stencil for the solution reconstruction, b) stencil for viscous flux integration.](image)

Viscous computations require the evaluation of the flow gradients at the quadrature points to all the faces of the control volume. The corresponding viscous MLS stencil is simply obtained by merging all of the MLS stencils of the first neighbors of the cell of interest (Fig. 1 on the left). From a practical point of view, the construction of the discretization stencil and the computation of the MLS shape function derivatives are done as a preprocessing step prior to the iterative procedure. As a consequence, the extra computational time relative to the third-order FV-MLS scheme compared to second order scheme is mainly due to the loop over the quadrature points required for the high-order flux integration.
3.3. **Flux integration on curved wall**

A common approach used to implement curved wall slip boundary conditions in a high-order finite volume solver is to place the Gauss quadrature points along the curved geometry and to modify the corresponding weights accordingly [27]. Furthermore, if the high-order reconstruction procedure requires the computation of control volumes moments to enforce the conservation of the mean, those terms must also be carefully evaluated along curved boundaries [32]. In this work, we propose to apply, in the context of a finite volume framework, the high-order accurate implementation of solid wall boundary conditions in curved geometry developed by Krivodonova and Berger [21] for the solution of the two-dimensional Euler equations using a discontinuous Galerkin method. The key issue of this approach is to account for the approximate or analytical [24] physical representation of the boundary normals. For boundary cells, we first introduce ghost states of the primitive variables \( \mathbf{q}_g = (\rho_g, u_g, v_g, p_g)^T \) at each integration points of the boundary edge according to

\[
\begin{align*}
\rho_g &= \rho \\
p_g &= p \\
u_g &= (\bar{n}_x^2 - \bar{n}_y^2) u - 2\bar{n}_x\bar{n}_y v \\
v_g &= (\bar{n}_x^2 - \bar{n}_y^2) v - 2\bar{n}_x\bar{n}_y u
\end{align*}
\]  

(22)

where \( \rho, u, v \) and \( p \) refer to the interior state, which is determined applying the MLS reconstruction at quadrature points. Notice that the unit normal vector to the physical boundary \( \mathbf{\hat{n}} = (\bar{n}_x, \bar{n}_y)^T \) differs from the unit normal vector to the computational boundary \( \mathbf{n} = (n_x, n_y)^T \). The last two equations in (22) are obtained by reflecting the velocity vector at each boundary integration point to the ghost state with respect to the tangential vector to the physical boundary [21]. Then, the Riemann problem at the boundary edge is solved in a similar manner as in (9) based on the normal of the computational straight boundary. A major benefit of this approach is that there is no need to recompute the Gauss integration data (point location and weight) according to the boundary curvature [31, 27], resulting in easier code implementation.

4. **Numerical applications**

In the following, we discuss in details the numerical results obtained using the present third-order FV-MLS solver. The correctness of the numerical
method is examined in subsection 4.1 by means of the analysis of the reconstruction procedure and flux integration. Assessment of the spatial accuracy for curved geometry is demonstrated in section 4.2. The robustness of the method for viscous flows is analyzed for a closed wake flow around a cylinder in section 4.3 and a laminar viscous flow past a NACA 0012 airfoil is investigated in section 4.4.

### 4.1. Accuracy assessment of the reconstruction procedure

In this section, the correctness of the implementation of the present MLS-based finite-volume method is assessed using a step-by-step examination of the high-order construction of the solution. To this end, we consider the following arbitrary smooth flow solution introduced by Ollivier-Gooch et al. [31]

\[
\begin{align*}
\rho(x, y) &= 1 + 0.1 \sin(\pi x) \sin(\pi y) \\
u(x, y) &= 0.1 \sin(\pi x) \cos(2\pi y) \\
v(x, y) &= 0.1 \cos(2\pi x) \sin(\pi y) \\
p(x, y) &= 5/7 + 0.1 \sin(2\pi x) \sin(2\pi y)
\end{align*}
\]

Computations are performed using a sequence of five uniformly refined triangular grids over a \([-0.5, 0.5] \times [-0.5, 0.5]\) square domain, with corresponding number of control volumes ranging from 180 to 50880.

First, we seek to compute the asymptotic convergence rates of the successive derivatives of the solution evaluated at cell centroids. All results were obtained using the full diffuse approximation (32) and a cubic spline kernel (13) with \(k = 0.65\). The \(L_2\)-norm of error in the derivatives of the density is computed according to

\[
er_{L_2} = \left( \frac{\sum_{i=1}^{N_{cv}} \Omega_i E_i^2}{\sum_{i=1}^{N_{cv}} \Omega_i} \right)^{\frac{1}{2}}
\]

For instance, the discrete error \(E_i\) of \(\partial \rho / \partial x\) of the \(i\)th cell reads

\[
E_i = \left| \left( \frac{\partial \rho}{\partial x} \right)_{i}^{\text{ML5}} - \left( \frac{\partial \rho}{\partial x} \right)_{i}^{\text{exact}} \right|
\]

where the MLS derivatives are given by Eq. (19). Figure 2a presents a log-log plot of the norm of error in the MLS derivatives of the density computed
using a fourth-order polynomial basis. As expected, the use of a \( r \)-th-order polynomial basis leads to a nominal order of \( r - s + 1 \) of a \( s \)-th-order gradient (Fig. 2a). The asymptotic convergence rates of the reconstruction of the density field at the quadrature points of the cell faces are 2.02 and 3 for the case of linear and quadratic polynomial basis respectively (Fig. 2b).

![Image](a) MLS derivatives (b) Taylor expansion

Figure 2: Verification of the accuracy of the MLS derivatives and the Taylor expansion reconstruction (all results were obtained using the full diffuse derivative approach (32) in conjunction with a quartic polynomial basis and a cubic spline kernel (13) with \( k = 0.65 \)).

Next, we check the ability of the present approach to satisfy the property of the mean conservation. From a practical point of view, we intend to show that the average of the exact density field over cell \( i \) is identical to the average of the reconstructed solution

\[
\bar{\rho}_i \equiv \frac{1}{|\Omega_i|} \int_{\Omega_i} \rho_i^{\text{exact}} \, dv = \frac{1}{|\Omega_i|} \int_{\Omega_i} N^T(x) \rho_{\Omega x} \, dv
\]

(26)

The control volume averages are computed using a tenth-order accurate quadrature rule [11]. Well-behaved convergence rates with slopes equal to \( p + 1 \) are observed for both linear and quadratic MLS reconstructions (Fig. 3a).

Finally, we show that the flux integration procedure is properly implemented by verifying that [31]

\[
\int_{|\Omega_i|} \nabla \cdot \vec{F}^{\text{exact}} \, dv = \sum_{\ell=1}^{n_f} \sum_{q=1}^{n_w} |\Gamma_{\ell}| w_q \vec{F}^{\text{num}}(Q_L, Q_R, \hat{n})
\]

(27)

where the integration of the analytical numerical fluxes is performed using quadrature rules for triangle which are exact for polynomials up to degree
Figure 3: Convergence rates of mean conservation and flux integration errors obtained using linear and quadratic polynomial basis for the computation of the derivatives of the MLS shape functions.

Figure 3b shows that the linear and quadratic polynomial basis give the expected flux integration error of order \( p + 2 \).

4.2. Inviscid subcritical flow around a circular cylinder

In order to investigate the solution accuracy of the present high-order FV-MLS solver for smooth flows, we compute the inviscid steady-state flow around a circular cylinder at Mach number \( M = 0.38 \). This test case is commonly used to examine the curved wall representation of high-order DG and FV methods. A grid refinement study is performed using a sequence of three refined O-type meshes with regular triangular cells as depicted in [24]. The finest grid shown in Fig. 4 consists in 128 points equally distributed in the circumferential direction and 32 points in the radial direction. Two additional \( 64 \times 16 \) and \( 32 \times 8 \) grids are obtained by coarsening the finest mesh in both directions. The farfield is situated at 40 chords away from the cylinder.

All computations, which are initialized using a uniform flow, are converged until the \( L_2 \) norm of the residuals falls below \( 10^{-8} \). The Mach isolines with step \( \Delta M = 0.038 \) is plotted in Fig. 5 for different polynomial order \( p \). The solution obtained using a linear reconstruction on the \( 64 \times 16 \) grid presents a nearly symmetric flow with respect to the coordinates axis (Fig. 5a). However an unphysical wake is visible downstream of the cylinder. A more accurate solution is obtained using a third-order scheme as shown on Fig. 5c. Results obtained for the \( 128 \times 32 \) mesh are presented on Fig. 5b.
Figure 4: $64 \times 16$ and $128 \times 32$ O-grids around a circular cylinder (the finest grid was computed according to [24]).

and Fig. 5d for $p = 1$ and $p = 2$ respectively.

Figure 6a presents a comparison of the distribution of the pressure coefficient $C_p = (p - p_\infty)/(0.5\rho_\infty||\vec{v}_\infty||)$ around the cylinder for the three different grids and for both linear and quadratic MLS reconstructions. Computations of coarse meshes show that the results are slightly improved using the $p = 2$ quadratic MLS scheme compared to the $p = 1$ solution with linear reconstruction. However, the flow remains asymmetrical the with respect to the longitudinal axis. The artificial wake is significantly reduced for the $64 \times 16$ medium grid, especially when the third-order scheme is employed. Finally, $p = 1$ and $p = 2$ solutions exhibits no visible differences for $128 \times 32$ grid.

A more detailed analysis of the flow can be conducted with the examination of the entropy production on the cylinder surface $\epsilon_{\text{ent}} = p/p_\infty (\rho/\rho_\infty)^\gamma - 1$. We remark on Fig. 6b that all numerical simulations exhibit a similar shape in the distribution of $\epsilon_{\text{ent}}$ over the cylinder, where highest values are always reached near the trailing edge. Both $h$-refinement and $p$-refinement results in substantial reductions in the entropy production $\epsilon_{\text{ent}}$. The maximum entropy production for the $p = 2$ solution on the $128 \times 32$ mesh is $1.1 \times 10^{-4}$.

The convergence rates relative to $p = 1$ and $p = 2$ FV-MLS computations are monitored on Fig. 7 by means of a log-log plot of the $L_2$ error norm of the entropy production $\epsilon_{\text{ent}}^{\text{err}}$ as a function of the grid size. It is clearly visible that the solution errors reach asymptotic rates of convergence close to the theoretical slopes.

Tab. 1 summarizes the order of convergence $\text{Ord}_{ab}$ of the method which
Figure 5: Computed Mach number isolines obtained by the FV-MLS method for a $M_\infty = 0.38$ flow past a circular cylinder.

is evaluated by comparison between the solution computed on two grids with characteristic mesh size $h_a$ and $h_b$

$$\text{Ord}_{ab} = \frac{\log(\text{err}_a) - \log(\text{err}_b)}{\log h_a - \log h_b}$$

(28)

where $\text{err}_{a,b,c}$ denotes either the $L_1$-norm or the $L_2$-norm of the error in the entropy production for the various computational grids of interest. We notice that the FV-MLS method leads to a full $O(h^{p+1})$ order of accuracy for both norms (Tab. 1). The level of discretization error always exceeds the expected theoretical order, as observed in previous studies based on discontinuous Galerkin methods [3, 21, 24].
Figure 6: Distributions of the pressure coefficient (a) and the entropy production (b) around the cylinder for both linear and quadratic MLS reconstruction.

Figure 7: $L_2$-norm of error in entropy as function of the number of cells for linear ($p = 1$) and quadratic ($p = 2$) MLS reconstruction.

Now, we analyze the effect of using diffuse (32), semi-diffuse (33) and full MLS derivatives (34) onto the accuracy of the FV-MLS solver. One may immediately observe on Tab. 2 that, for the same order of the polynomial basis, the global accuracy of the method is not strongly affected by the three approaches. We also note that, neglecting the first derivatives of $C(x)$ ( $p = 1$; case A in Tab. 2) gives acceptable results compared to orders of accuracy obtained using the full $p = 1$ MLS derivatives (case B in Tab. 2). Results obtained for $p = 2$ confirm that the use of the semi-diffuse approximation not only increases the formal order of the numerical FV-MLS
Table 1: \( L_1 \) and \( L_2 \) norms of error in entropy production and orders of accuracy of the FV-MLS solver for the inviscid flow around a cylinder at \( M_\infty = 0.38 \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>Mesh size</th>
<th>( L_1 )-error</th>
<th>order</th>
<th>( L_2 )-error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 a</td>
<td>32 \times 6</td>
<td>1.7434E-03</td>
<td>-</td>
<td>4.2840E-03</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>64 \times 16</td>
<td>2.2145E-04</td>
<td>2.98</td>
<td>6.6611E-04</td>
<td>2.68</td>
</tr>
<tr>
<td></td>
<td>128 \times 32</td>
<td>3.0371E-05</td>
<td>2.87</td>
<td>1.0275E-04</td>
<td>2.69</td>
</tr>
<tr>
<td>2 a</td>
<td>32 \times 6</td>
<td>3.0923E-03</td>
<td>-</td>
<td>7.0105E-03</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>64 \times 16</td>
<td>2.5902E-04</td>
<td>3.57</td>
<td>6.4485E-04</td>
<td>3.44</td>
</tr>
<tr>
<td></td>
<td>128 \times 32</td>
<td>1.6315E-05</td>
<td>3.98</td>
<td>4.7838E-05</td>
<td>3.75</td>
</tr>
</tbody>
</table>

Table 2: Influence of different approximations of the MLS shape function derivatives on the formal order of the method (the order of accuracy is evaluated by comparison of the entropy error between grid B (64 \times 16) and grid C (128 \times 32).

<table>
<thead>
<tr>
<th>case</th>
<th>derivatives</th>
<th>( p )</th>
<th>( L_1 )-error</th>
<th>( L_2 )-error</th>
<th>Ord(^p_1 )</th>
<th>Ord(^p_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>diffuse</td>
<td>1</td>
<td>3.9395E-05</td>
<td>1.1868E-05</td>
<td>2.75</td>
<td>2.65</td>
</tr>
<tr>
<td>B</td>
<td>full</td>
<td>1</td>
<td>3.0371E-05</td>
<td>1.0275E-05</td>
<td>2.87</td>
<td>2.69</td>
</tr>
<tr>
<td>C</td>
<td>diffuse</td>
<td>2</td>
<td>1.6315E-05</td>
<td>4.7838E-05</td>
<td>3.98</td>
<td>3.75</td>
</tr>
<tr>
<td>D</td>
<td>semi-diffuse</td>
<td>2</td>
<td>1.5859E-05</td>
<td>4.7098E-05</td>
<td>4.02</td>
<td>3.76</td>
</tr>
<tr>
<td>E</td>
<td>full</td>
<td>2</td>
<td>1.2881E-05</td>
<td>3.6558E-05</td>
<td>3.90</td>
<td>3.87</td>
</tr>
</tbody>
</table>

Fig. 8 shows that the distribution of the pressure coefficient along the surface do not strongly differ when diffuse or full derivatives are employed. However, it is interesting to note that the entropy production on the wall decreases when the full derivatives are computed, leading to a reduction in the maximum entropy production by a factor ranging to 17\% for the 64 \times 16 grid up to 25\% for the 128 \times 32 grid.

Our conclusion is that this study confirms that the diffuse approximation of the derivatives do not deteriorate the \( O(h^{p+1}) \) order of converge on smooth inviscid solutions as it was pointed out in [18]. However, for higher-order computations (e.g. \( p > 3 \)) less dissipative results should be obtained using the semi-diffuse approach with moderate extra coding effort compared to the full derivative approach.

Finally, we intend to evaluate how the present solver is sensitive to the high-order representation of the boundary flux integrals. To this end, we
Figure 8: Effect of full (33) and diffuse (32) approximations of the shape function derivatives on the pressure coefficient (left) and entropy (right) distributions around the cylinder with $p = 2$ quadratic reconstruction.

consider the previous third-order FV-MLS scheme with the diffuse derivative approximation as reference (namely case C in Tab. 2). As expected, we remark from Tab. 3 that the use of a straight representation of the wall normals (case A) results in a loss of accuracy compared to the case based on an accurate reconstruction of physical normals (case B).

<table>
<thead>
<tr>
<th>case</th>
<th>wall normal</th>
<th>$L_1$-error</th>
<th>Order</th>
<th>$L_2$-error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>straight</td>
<td>2.7771E−03</td>
<td>-</td>
<td>6.3479E−03</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.0751E−04</td>
<td>2.74</td>
<td>5.5325E−04</td>
<td>3.52</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.9717E−05</td>
<td>3.39</td>
<td>7.3874E−05</td>
<td>2.90</td>
</tr>
<tr>
<td>B</td>
<td>physical</td>
<td>3.0923E−03</td>
<td>-</td>
<td>7.0105E−03</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.5902E−04</td>
<td>3.57</td>
<td>6.4485E−04</td>
<td>3.44</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.6315E−05</td>
<td>3.98</td>
<td>4.7838E−05</td>
<td>3.75</td>
</tr>
</tbody>
</table>

Table 3: Effect of different wall boundary flux representations on the accuracy of the third-order FV-MLS method.

More surprisingly, we remark that the third-order nominal accuracy is maintained when the wall normal are computed from the computational geometry. Moreover, minor differences are observed between these two representations of wall normals on the Mach number isolines computed for a 64 × 16 grid (Fig. 9).

This observation suggests that, for the present test case, the high-order
FV-MLS solver is not as sensitive to improper solid wall representation as the discontinuous Galerkin method [3, 21, 9, 10, 24] which may exhibit unphysical vortex shedding when straight-sided high-order elements are employed. One possible reason for the bigger DGM and SVM sensitivities to the curved boundary is the nature of the discretization scheme. In DGM, we use high-order elements, whereas high-order FVM employ zero-order elements in conjunction with a reconstruction stencil. When high-order elements are used, variable values at integration points are evaluated using nodal values over a single straight-sided element (Fig. 10). In an isoparametric approach this implies the assumption of straight lines in the geometry. Therefore, this piecewise-linear boundary representation may lead to unphysical or unsteady vortex shedding for the problem of an inviscid flow past a circular cylinder. One possible way to avoid this problem is the use of boundary fitted high-order curved elements with corresponding computational expenses due to the non-linear mapping of curved elements onto straight-sided elements.

On the other hand, the high-order FV-MLS method uses neighbor nodes to compute the value of the variables at integration points. Moreover, we use a set of ghost cells to improve the accuracy of the computations of derivatives near the wall. In the figure, we can see that the ghost cells give a better approximation of the geometry than straight-sided elements of DGM,
resulting in a less sensitive approach. Note that no assumption is made on the geometry of the boundary, other than the direction of the normals. This assumption seems to be weaker than that of the isoparametric approach. As shown in the paper, the accuracy of the FV-MLS approach is increased when the representation of the boundary normals used to define a ghost state at quadrature points is based on unit normal to the physical boundary and not the straight-sided face normal.

In finite volume discretization, using a quadratic reconstruction of the flow variables at wall boundary is of the utmost importance. Table 4 highlights this importance in order to keep the error in entropy production as small as possible. A third-order FV-MLS scheme with quadratic reconstruction of the Riemann states is used for interior cells. We also notice that a genuinely third-order accurate method can be obtained using a linear reconstruction step. However, the use of a first order accurate representation of the wall boundary conditions lead to a second-order nominal accuracy.

This remark is illustrated on Fig. 11a which shows that a nonphysical wake appears at the trailing edge of the cylinder for improper boundary conditions. On the contrary, the flow patterns remains completely symmetrical when a quadratic reconstruction of physical normals is employed (Fig. 11b).
### Table 4: \( L_1 \) and \( L_2 \) norms of error in entropy production and orders of accuracy of the FV-MK solver for the inviscid flow problem for constant, linear and quadratic reconstruction of the Riemann states at wall edges (\( p_{bc} \) denotes the order of the polynomial basis in the computation of the MLS shape function derivatives).

<table>
<thead>
<tr>
<th>case</th>
<th>wall reconstruction</th>
<th>( p_{bc} )</th>
<th>( L_1 )-error</th>
<th>Order</th>
<th>( L_2 )-error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>constant</td>
<td>0</td>
<td>5.0204E-03</td>
<td>-</td>
<td>1.1925E-02</td>
<td>-</td>
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<td></td>
<td></td>
<td></td>
<td>1.0728E-03</td>
<td>2.22</td>
<td>3.6569E-03</td>
<td>1.70</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.5466E-04</td>
<td>2.07</td>
<td>1.2632E-03</td>
<td>1.53</td>
</tr>
<tr>
<td>B</td>
<td>linear</td>
<td>1</td>
<td>3.1717E-03</td>
<td>-</td>
<td>7.2204E-03</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.8543E-04</td>
<td>3.47</td>
<td>7.4420E-04</td>
<td>3.27</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.1016E-05</td>
<td>3.76</td>
<td>7.2939E-05</td>
<td>3.35</td>
</tr>
<tr>
<td>C</td>
<td>quadratic</td>
<td>2</td>
<td>3.0923E-03</td>
<td>-</td>
<td>7.0105E-03</td>
<td>-</td>
</tr>
<tr>
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<td></td>
<td></td>
<td>2.5902E-04</td>
<td>3.57</td>
<td>6.4485E-04</td>
<td>3.44</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.6315E-05</td>
<td>3.98</td>
<td>4.7838E-05</td>
<td>3.75</td>
</tr>
</tbody>
</table>

Figure 11: (a) Constant (\( p_{bc}=0 \)) and (b) quadratic (\( p_{bc}=2 \)) reconstructions of the flow variables at solid edges (Riemann states at interior cells are computed using a third-order quadratic MLS reconstruction, the grid size is 128 \( \times \) 32).

### 4.3. Steady closed wake flow around a cylinder

In this section, we consider a steady closed wake flow around a circular cylinder as the first test case for the validation of our high-order FV-MLS method for viscous flow problems. The free stream Mach number is \( M_\infty = 0.1 \) and the diameter of the cylinder is set to \( D = 1 \). An unstructured regular
grid, with $128 \times 64$ triangular elements was employed for both second- and third-order FV-MLS computations (Fig. 12). The outer boundary is situated at a distance from the cylinder equal to $20D$. No-slip and adiabatic boundary conditions are applied at solid walls.

Figure 12: Unstructured triangular grid and near-wall closed view for the computation of laminar viscous flows past a circular cylinder.

Figure 13 shows the streamlines around the cylinder which are computed using the third-order FV-MLS scheme at $Re = 40$. We notice that the present numerical method is able to predict the expected wake pattern with a pair of symmetric contra-rotating vortices.

Figure 14 plots the size of the wake length $L$ as a function of the Reynolds number which ranges from $Re = 10$ to $Re = 40$. FV-MLS computations present a good agreement with the data given in [5]. We notice however, that better predictions are obtained using the third-order accurate MLS discretization compared to the second-order scheme, in particular, as far as the growth rate of the wake length is concerned.
Next, we compute the geometrical parameters of the closed wake by means of the length of the circulation region, the coordinates of the vortex center \((a, b)\) and the separation angle \(\theta_s\) at the cylinder surface (Fig. 13). These
parameters are compared in table 5 with previous experimental [5] and numerical [16, 23] studies for \( Re = 20 \) and \( Re = 40 \). We observe that the values obtained using the present numerical method are in a comparable range to those published in the literature for both second- and third-order MLS schemes. Considering [23] as the reference results for \( Re = 40 \), we obtain a close agreement (about 1.4%) on the computation of \( \theta_s \). In general, the third-order FV-MLS results are more accurate as shown by the prediction on the wake length whose difference is reduced from 5.5% for \( p = 1 \) to 0.9% for \( p = 2 \). We notice however that both schemes underestimate the longitudinal coordinate of the vortex center \( a \) by about 10% compared to [23] but the computation of \( b \) is satisfactory (tab. 5).

<table>
<thead>
<tr>
<th>Reynolds number</th>
<th>20</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L/R )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ref. [5]</td>
<td>0</td>
<td>4.26</td>
</tr>
<tr>
<td>Ref. [16]</td>
<td>1.842</td>
<td>4.49</td>
</tr>
<tr>
<td>Ref. [23]</td>
<td>0</td>
<td>4.34</td>
</tr>
<tr>
<td>FV-MLS (( p = 1 ))</td>
<td>1.713</td>
<td>4.1</td>
</tr>
<tr>
<td>FV-MLS (( p = 2 ))</td>
<td>1.84</td>
<td>4.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>((a/D, b/2D))</th>
<th>Ref. [5]</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref. [23]</td>
<td></td>
<td>(1.52, 1.19)</td>
</tr>
<tr>
<td>FV-MLS (( p = 1 ))</td>
<td>(0.67, 0.82)</td>
<td>(1.34, 1.14)</td>
</tr>
<tr>
<td>FV-MLS (( p = 2 ))</td>
<td>(0.70, 0.85)</td>
<td>(1.34, 1.17)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \theta_s ) (deg)</th>
<th>Ref. [5]</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref. [16]</td>
<td>42.96</td>
<td>52.84</td>
</tr>
<tr>
<td>Ref. [23]</td>
<td>0</td>
<td>53.13</td>
</tr>
<tr>
<td>FV-MLS (( p = 1 ))</td>
<td>41.83</td>
<td>52.39</td>
</tr>
<tr>
<td>FV-MLS (( p = 2 ))</td>
<td>43.73</td>
<td>52.71</td>
</tr>
</tbody>
</table>

Table 5: Comparison of geometrical parameters with experimental [5] and numerical [16, 23] studies for a steady wake flow around a cylinder at \( Re = 20 \) and \( Re = 40 \).

Finally, the aerodynamical parameters are reported in table 6 in terms of drag coefficient \( C_d \), front \( C_p(\pi) \) and back \( C_p(0) \) pressure coefficients. Significant improvements are obtained using the highest-order scheme whose predictions are reasonably good compared to previous results at \( Re = 20 \).
and $Re = 40$.

<table>
<thead>
<tr>
<th></th>
<th>Reynolds number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>20</td>
</tr>
<tr>
<td>$C_d$</td>
<td>Ref. [16]</td>
</tr>
<tr>
<td></td>
<td>Ref. [28]</td>
</tr>
<tr>
<td></td>
<td>FV-MLS ($p = 1$)</td>
</tr>
<tr>
<td></td>
<td>FV-MLS ($p = 2$)</td>
</tr>
<tr>
<td>$C_p(\pi)$</td>
<td>Ref. [16]</td>
</tr>
<tr>
<td></td>
<td>Ref. [28]</td>
</tr>
<tr>
<td></td>
<td>FV-MLS ($p = 1$)</td>
</tr>
<tr>
<td></td>
<td>FV-MLS ($p = 2$)</td>
</tr>
<tr>
<td>$C_p(0)$</td>
<td>Ref. [16]</td>
</tr>
<tr>
<td></td>
<td>Ref. [28]</td>
</tr>
<tr>
<td></td>
<td>FV-MLS ($p = 1$)</td>
</tr>
<tr>
<td></td>
<td>FV-MLS ($p = 2$)</td>
</tr>
</tbody>
</table>

Table 6: Comparison of aerodynamic parameters with previous studies [16, 28] for a steady wake flow around a cylinder at $Re = 20$ and $Re = 40$.

4.4. Viscous flow past a NACA 0012 airfoil

The last configuration concerns a laminar subsonic flow past a NACA 0012 airfoil at zero angle of attack. Flow conditions correspond to a free stream Mach number $M_\infty = 0.5$ and $Re_\infty = 5000$. This problem is widely used for validation purpose of second- and higher-order Navier-Stokes solvers [20, 19, 15, 25, 26, 34, 38]. This is a challenging test case because the flow separates near the trailing and induces a small recirculation region which extends in the wake [15, 25, 26, 34, 38] that is difficult to reproduce accurately.

As shown in Fig. 15, we considered an unstructured hybrid grid with rectangular control volumes near the solid wall and with triangular elements away from the profile. The total number of elements is 29107 and 450 cells are distributed around the surface of the profile. The outer boundary is placed at a distance of 20 chords away from the airfoil.
Figure 15: Closed-view of the hybrid computational grid near the NACA 0012 airfoil.

Figure 16a shows the contours of the Mach number computed using the third-order FV-MLS scheme. The pattern of the flow separation near the trailing edge and the recirculation zone are detailed by means of streamlines in Fig. 16b. We remark that the present numerical approach succeeds in predicting symmetrical counter rotating vortices which extends to the wake.

The distributions of the pressure coefficient $C_p$ and the skin friction coefficient $C_f$ over the airfoil surface are plotted in Fig. 17. The agreement of both second- and third-order FV-MLS schemes with the results obtained using a fourth-order spectral volume approach [38] is very satisfactory (Fig. 17a). The profiles of $C_f$ computed using linear and quadratic polynomial basis are nearly identical (Fig. 17b). In particular, there are no major differences in the prediction of the location of the separation region (Fig. 17c). This is in contrast with the observations given in [38] based on spectral volumes on unstructured grids.
Figure 16: (a) Contours of the Mach number over the NACA 0012 airfoil for $M_\infty = 0.5$ and $Re_\infty = 5000$ obtained using the third-order FV-MLS scheme (b) closed view of the small circulation bubble in the near wake.
Figure 17: (a) Comparison of the computed pressure coefficient using second- and third-order FV-MLS approaches and a fourth-order spectral volume method [38] (b) Distribution of the skin friction coefficient along the NACA 0012 airfoil for $M_\infty = 0.5$ and $Re_\infty = 5000$ (c) Closed view of the skin friction coefficient in the rear part of the airfoil. It is seen the slightly oscillatory shape of the $C_f^*$ obtained with the second order method.
However, we remark that the profile of $C_f$ resulting from the second-order FV-MLS scheme presents a slightly oscillatory shape contrary to those obtained using a higher-order scheme.

The drag coefficient and the location of the separation point are compared with previous work [15, 25, 26, 34] in Table 7. Although most of these studies were performed on structured grids or on finer computational grids, we notice that FV-MLS simulations give similar results for all parameters. The computed values given by the third-order scheme agree with those of [34] obtained on a $512 \times 128$ structured grid agree within 1.35% for the location of the separation point and within 0.6% and 1.3% for the viscous and pressure $C_d$ respectively.

Table 7: Comparison of the drag coefficient and the location of the separation point between the FV-MLS method and previous numerical studies for a laminar flow around a NACA 0012 airfoil at $M_\infty = 0.5$ and $Re_\infty = 5000$.

<table>
<thead>
<tr>
<th></th>
<th>grid</th>
<th>$C_d$ (viscosity)</th>
<th>$C_d$ (pressure)</th>
<th>separation point</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref. [15]</td>
<td>$320 \times 128$</td>
<td>0.0321</td>
<td>0.0221</td>
<td>0.824</td>
</tr>
<tr>
<td>Ref. [25]</td>
<td>$320 \times 64$</td>
<td>0.0332</td>
<td>0.0229</td>
<td>0.814</td>
</tr>
<tr>
<td>Ref. [26]</td>
<td>37399</td>
<td>0.0323</td>
<td>0.0224</td>
<td>0.778</td>
</tr>
<tr>
<td>Ref. [34]</td>
<td>$512 \times 128$</td>
<td>0.0330</td>
<td>0.0224</td>
<td>0.814</td>
</tr>
<tr>
<td>FV-MLS ($p = 1$)</td>
<td>29107</td>
<td>0.0318</td>
<td>0.0222</td>
<td>0.803</td>
</tr>
<tr>
<td>FV-MLS ($p = 2$)</td>
<td>29107</td>
<td>0.0328</td>
<td>0.0227</td>
<td>0.808</td>
</tr>
</tbody>
</table>

5. Concluding remarks

In this paper, we have studied some properties of a third-order finite-volume moving least-squares method (FV-MLS) for the solution of the two-dimensional Euler and Navier-Stokes equations on unstructured triangular grids. Viscous fluxes are directly computed at integration points. MLS stencil of viscous fluxes is simply obtained from the union of MLS stencils of the cells sharing the face. Numerical tests performed on an inviscid flow around a circular cylinder show that the method exhibits the expected formal order of accuracy in both the $L_1$- and $L_2$-norms depending on the order of the polynomial basis. Surprisingly, we have also observed that, differently from Discontinuous Galerkin discretization, the present FV-MLS formulation is not very sensitive to high-order accurate representations of curved
wall boundary. This is a remarkable feature of the numerical method. In addition, we have performed a study of the influence of using diffuse derivatives, instead of the full derivatives. We have observed a loss of accuracy, although the global order of convergence is maintained. A semi-diffuse method of computation of the derivatives is proposed, as a compromise between accuracy and computational cost. The FV-MLS scheme was successfully applied to two benchmark viscous flow configuration and results demonstrate the relevant capabilities of the use of moving least-squares approximations in the context of high-order finite volume methods.

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References


APPENDIX: Computation of the MLS shape function derivatives

Here we give the expression of the full derivatives of the shape function for quadratic reconstruction. Recall the expression of the first derivatives of $N^T(x)$ given in (20)

$$\frac{\partial N^T(x)}{\partial x} = \frac{\partial p^T(x)}{\partial x} C(x) + p^T(x) \frac{\partial C(x)}{\partial x}$$

(29)

the expression of the second derivatives of $N^T(x)$ are

$$\frac{\partial^2 N^T(x)}{\partial x^2} = \frac{\partial^2 p^T(x)}{\partial x^2} C(x) + 2 \frac{\partial p^T(x)}{\partial x} \frac{\partial C(x)}{\partial x} + p(x) \frac{\partial^2 C(x)}{\partial x^2}$$

$$\frac{\partial^2 N^T(x)}{\partial x \partial y} = \frac{\partial^2 p^T(x)}{\partial x \partial y} C(x) + \frac{\partial p^T(x)}{\partial x} \frac{\partial C(x)}{\partial y} + \frac{\partial p^T(x)}{\partial y} \frac{\partial C(x)}{\partial x} + p^T(x) \frac{\partial^2 C(x)}{\partial x \partial y}$$

(30)
where $\partial C(x)/\partial x$ is given by (21)

$$\frac{\partial C(x)}{\partial x} = C(x) W^{-1}(x) \frac{\partial W(x)}{\partial x} (I - p^T(x)C(x)) \quad (31)$$

The diffuse derivatives [6, 7] are obtained by neglecting all derivatives of $C(x)$ in (30)

$$\begin{align*}
\frac{\partial^2 N^T(x)}{\partial x^2} & \approx \frac{\partial^2 p^T(x)}{\partial x^2} C(x) \\
\frac{\partial^2 N^T(x)}{\partial x \partial y} & \approx \frac{\partial^2 p^T(x)}{\partial x \partial y} C(x) + \frac{\partial p^T(x)}{\partial x} \frac{\partial C(x)}{\partial x} + \frac{\partial p^T(x)}{\partial y} \frac{\partial C(x)}{\partial y} \quad (32)
\end{align*}$$

However, since the first derivatives of $C(x)$ are already employed in (29), we can used the following semi-diffuse approximation without extra coding effort

$$\begin{align*}
\frac{\partial^2 N^T(x)}{\partial x^2} & \approx \frac{\partial^2 p^T(x)}{\partial x^2} C(x) + 2 \frac{\partial p^T(x)}{\partial x} \frac{\partial C(x)}{\partial x} + \frac{\partial p^T(x)}{\partial x} \frac{\partial C(x)}{\partial x} \quad (33)
\end{align*}$$

Finally, the full second-order derivatives of $N^T$ are completely determined by

$$\begin{align*}
\frac{\partial^2 C(x)}{\partial x^2} & = \frac{\partial C(x)}{\partial x} W^{-1}(x) \frac{\partial W(x)}{\partial x} (I - p^T(x)C(x)) + C(x) W^{-1}(x) \frac{\partial^2 W(x)}{\partial x^2} (I - p^T(x)C(x)) - C(x) W^{-1}(x) \frac{\partial W(x)}{\partial x} \frac{\partial W(x)}{\partial x} (I - p^T(x)C(x)) - C(x) W^{-1}(x) \frac{\partial W(x)}{\partial x} \frac{\partial C(x)}{\partial x} \quad (34)
\end{align*}$$

$$\begin{align*}
\frac{\partial^2 C(x)}{\partial x \partial y} & = \frac{\partial C(x)}{\partial y} W^{-1}(x) \frac{\partial W(x)}{\partial x} (I - p^T(x)C(x)) + C(x) W^{-1}(x) \frac{\partial^2 W(x)}{\partial x \partial y} (I - p^T(x)C(x)) - C(x) W^{-1}(x) \frac{\partial W(x)}{\partial y} \frac{\partial W(x)}{\partial x} (I - p^T(x)C(x)) - C(x) W^{-1}(x) \frac{\partial W(x)}{\partial y} \frac{\partial C(x)}{\partial x} \quad (35)
\end{align*}$$
Some important properties of a finite-volume moving least-squares method are studied. A particular attention is paid to the computation of shape function derivatives. High-order schemes for compressible flows on unstructured grid are then developed. Straight representation of wall normals does not induce important losses of accuracy. Accuracy and robustness are assessed for both inviscid and viscous flows.