Metrizable TAP, HTAP and STAP groups

X. Domínguez\textsuperscript{a}, V. Tarieladze\textsuperscript{b}

\textsuperscript{a}Dept. de Métodos Matemáticos y de Representación, Universidad de A Coruña, Spain
\textsuperscript{b}Niko Muskhelishvili Institute of Computational Mathematics, Tbilisi, Georgia

Abstract

In a recent paper by D. Shakhmatov and J. Spěvák [Group-valued continuous functions with the topology of pointwise convergence, Topology Appl. 157 (2010) 1518–1540] the concept of a TAP group is introduced and it is shown in particular that NSS groups are TAP. We define the classes of STAP and HTAP groups and show that in general one has the inclusions NSS $\subset$ STAP $\subset$ HTAP $\subset$ TAP. We show that metrizable STAP groups are NSS and that Weil-complete metrizable TAP groups are NSS as well. We prove that an abelian TAP group is HTAP, while, as recently proved by D. Dikranjan and the above mentioned authors, there are nonabelian metrizable TAP groups which are not HTAP. A remarkable characterization of pseudocompact spaces obtained in the above mentioned paper asserts: a Tychonoff space $X$ is pseudocompact if and only if $C_p(X, \mathbb{R})$ has the TAP property. We show that for no infinite Tychonoff space $X$, the group $C_p(X, \mathbb{R})$ has the STAP property. We also show that a metrizable locally balanced topological vector group is STAP iff it does not contain a subgroup topologically isomorphic to $\mathbb{Z}^{(\mathbb{N})}$.

Keywords: topological group, metrizable group, NSS group, multipliable sequence, summable sequence, topological vector group

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Email addresses: xabier.dominguez@udc.es (X. Domínguez),
vajatarieladze@yahoo.com (V. Tarieladze)

\textsuperscript{1}This paper is dedicated to Professor Dikran Dikranjan on the occasion of his 60th birthday.
1. Introduction

In [8, Definition 4.1] a subset \(A\) of a topological group \(G\) is called absolutely productive in \(G\) provided that, for every injection \(a : \mathbb{N} \to A\) and each mapping \(z : \mathbb{N} \to \mathbb{Z}\), the sequence \(\left(\prod_{k=1}^{n} a(k)^{z(k)}\right)_{n \in \mathbb{N}}\) converges to some \(g \in G\).

In [8, Definition 4.5] a topological group \(G\) is called TAP (an abbreviation for “trivially absolutely productive”) if every absolutely productive set in \(G\) is finite.

For a topological group \(G\) we call a (not necessarily injective) sequence \((g_{n})_{n \in \mathbb{N}}\) of elements of \(G\)

- unconditionally hyper-multipliable in \(G\) provided that for each mapping \(z : \mathbb{N} \to \mathbb{Z}\) and for each bijective \(\sigma : \mathbb{N} \to \mathbb{N}\) the sequence \(\left(\prod_{k=1}^{n} g_{\sigma(k)}^{z(k)}\right)_{n \in \mathbb{N}}\) converges to some \(g \in G\).

- hyper-multipliable in \(G\) provided that for each mapping \(z : \mathbb{N} \to \mathbb{Z}\) the sequence \(\left(\prod_{k=1}^{n} g_{k}^{z(k)}\right)_{n \in \mathbb{N}}\) converges to some \(g \in G\).

- hyper-converging in \(G\) provided that for each mapping \(z : \mathbb{N} \to \mathbb{Z}\) the sequence \(\left(g_{z(n)}^{n}\right)_{n \in \mathbb{N}}\) converges to the neutral element \(e\) of \(G\).

In terms of unconditionally hyper-multipliable sequences the definition of a TAP group can be formulated as follows: A topological group \(G\) is TAP iff no injective sequence \((g_{n})_{n \in \mathbb{N}}\) of elements of \(G\) is unconditionally hyper-multipliable in \(G\). Motivated by this observation, we call a topological group \(G\)

HTAP (an abbreviation for “Hyper TAP”) iff no injective sequence \((g_{n})_{n \in \mathbb{N}}\) of elements of \(G\) is hyper-multipliable in \(G\).

STAP (an abbreviation for “Strictly TAP”) iff no injective sequence \((g_{n})_{n \in \mathbb{N}}\) of elements of \(G\) is hyper-converging in \(G\).

It is clear that if \(G\) is HTAP then it is TAP; the converse is true provided either \(G\) is abelian or \(G\) is Weil-complete metrizable (see Theorems 3.6 and 3.11). It was not known for a while whether a non-abelian TAP group is necessarily HTAP. Only quite recently in [4, Theorem 14.1] the existence of a metrizable separable TAP group which is not a HTAP group was established.
In [8] it is proved that NSS groups are TAP and that the converse statement fails in general (see [8, Theorem 4.9 and Remark 4.11]). We show that NSS groups are in fact STAP (Theorem 3.6(a)) and that the converse statement is true in metrizable case (Theorem 3.9). We prove also that for Weil-complete metrizable groups NSS and TAP properties are equivalent (Theorem 3.11).

A remarkable characterization of pseudocompact spaces obtained in [8, Theorem 6.5] says: a Tychonoff space $X$ is pseudocompact if and only if $C_p(X, \mathbb{R})$ has the TAP property. We show that for no infinite Tychonoff space $X$, the group $C_p(X, \mathbb{R})$ has the STAP property (Theorem 4.3).

In Section 2 the hyper-multiplicable, hyper-converging and related sequences are discussed. Remark 2.16 contains a negative answer to [4, Question 15.5]. Section 3 deals with general TAP, STAP and NSS groups. In Section 4 the case of $C_p(X, \mathbb{R})$ is considered.

In Sections 5 and 6, inspired by results from [1, 7], we introduce the class MMP of metrizable abelian groups (which includes the class of all metrizable topological vector spaces over $\mathbb{R}$) and show that if $G \in$ MMP is a complete group, then $G$ is a NSS-group iff $G$ does not contain a subgroup topologically isomorphic to $\mathbb{Z}^\mathbb{N}$ (Theorem 6.10).

**Notation.** In what follows $G$ will stand for a Hausdorff topological group with multiplication $\cdot$ and with neutral element $e$. In case of abelian groups with the operation $+$, the neutral element will be denoted by $0$. The symbol $\mathcal{N}(G)$ will denote the family of all neighborhoods of $e$ in $G$.

We say that $G$ is a NSS (No Small Subgroups) group if $G$ has a neighborhood of $e$ containing no nontrivial subgroups of $G$.

If $a$ is an element of a group $G$, we will denote by $\langle a \rangle$ the subgroup of $G$ generated by $a$. If $a$ is an element of a real vector space $E$, we will denote by $\mathbb{R} a$ the linear subspace of $E$ generated by $a$.

$\mathbb{R}_+$ denotes the set of all nonnegative real numbers. $\mathbb{N}$ is the set of strictly positive integers endowed with the usual order; $\mathcal{F}(\mathbb{N})$ is the set of finite subsets of $\mathbb{N}$ endowed with the partial order $\subset$, the inclusion.

For a fixed sequence $(g_n)_{n \in \mathbb{N}}$ of elements of $G$ and a set $\Delta \in \mathcal{F}(\mathbb{N})$ we define the symbol $\prod_{n \in \Delta} g_n$ as follows: If $\Delta = \emptyset$, then $\prod_{n \in \Delta} g_n = e$. If $\text{card}(\Delta) = k \geq 1$ and $\Delta = \{n_1, \ldots, n_k\}$, where $n_1 < \cdots < n_k$, then $\prod_{n \in \Delta} g_n = g_{n_1} \cdots g_{n_k}$. We also denote by $\Delta_n$ the subset $\{1, 2, \ldots, n\} \subset \mathbb{N}$, for every $n \in \mathbb{N}$, and write $\prod_{k \in \Delta_n} g_k$ simply as $\prod_{k=1}^n g_k$. We will replace the symbol $\prod$ with $\sum$ whenever we use additive notation.

In the sequel “Cauchy sequence (net)” will mean “left Cauchy sequence
A topological group $G$ is called sequentially Weil-complete (Weil-complete) if every $G$-valued Cauchy sequence (Cauchy net) converges in $G$.

For any nonempty family $(G_i)_{i \in I}$ of topological groups we will denote by $\prod_{i \in I} G_i$ the topological product of the family. If $G_i = G$ for every $i \in I$ we will write $G^I$ instead of $\prod_{i \in I} G_i$. We denote by $G(I)$ the subgroup of $G^I$ formed by the elements $(g_i)_{i \in I} \in \prod_{i \in I} G_i$ such that $g_i = e$ for all but finitely many $i \in I$.

2. Hyper-multipliable and related sequences

**Definition 2.1.** We call a sequence $(g_n)_{n \in \mathbb{N}}$ of elements of $G$

- simply multipliable if the sequence $(\prod_{k=1}^{n} g_k)_{n \in \mathbb{N}}$ converges to some $g \in G$.

- hyper-convergent if for every sequence $(m_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{Z}$ the sequence $(g_n^{m_n})_{n \in \mathbb{N}}$ converges to $e$.

- (cf. [2, Appendice II, Définition 1]) multipliable in Bourbaki’s sense (B-multipliable) if the net $(\prod_{n \in \Delta} g_n)_{\Delta \in \mathcal{F}(\mathbb{N})}$ converges to some $g \in G$.

- super-multipliable (or hereditarily multipliable) in $G$ if for every sequence $(m_k)_{k \in \mathbb{N}}$ of elements of $\{0,1\} \subset \mathbb{Z}$ the sequence $(\prod_{k=1}^{n} g_k^{m_k})_{n \in \mathbb{N}}$ converges to some $g \in G$ (equivalently, for every strictly increasing sequence of natural numbers $(i_n)_{n \in \mathbb{N}}$ the sequence $(g_{i_n})_{n \in \mathbb{N}}$ is simply multipliable).

- unconditionally super-multipliable (or unconditionally hereditarily multipliable) in $G$ if for every sequence $(m_k)_{k \in \mathbb{N}}$ of elements of $\{0,1\} \subset \mathbb{Z}$ and for each bijective $\sigma : \mathbb{N} \to \mathbb{N}$ the sequence $(\prod_{k=1}^{n} g_k^{m_k})_{n \in \mathbb{N}}$ converges to some $g \in G$ (equivalently, if for every bijective $\sigma : \mathbb{N} \to \mathbb{N}$ the sequence $(g_{\sigma(n)})_{n \in \mathbb{N}}$ is super-multipliable).

- hyper-multipliable in $G$ if for every sequence $(m_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{Z}$ the sequence $(\prod_{k=1}^{n} g_k^{m_k})_{n \in \mathbb{N}}$ converges to some $g \in G$.

- unconditionally hyper-multipliable in $G$ if for every sequence $(m_k)_{k \in \mathbb{N}}$ of elements of $\mathbb{Z}$ and for each bijective $\sigma : \mathbb{N} \to \mathbb{N}$ the sequence $(\prod_{k=1}^{n} g_k^{m_k})_{n \in \mathbb{N}}$ converges to some $g \in G$ (equivalently, if for every bijective $\sigma : \mathbb{N} \to \mathbb{N}$ the sequence $(g_{\sigma(n)})_{n \in \mathbb{N}}$ is hyper-multipliable).
• eventually neutral if the set \( \{ n \in \mathbb{N} : g_n \neq e \} \) is finite.

In case of abelian groups with the operation \(+\), instead of the terms “simply multipliable”, “B-multipliable”, “(unconditionally) super-multipliable”, “(unconditionally) hyper-multipliable”, we shall use the terms “simply summable”, “B-summable”, “(unconditionally) super-summable”, “(unconditionally) hyper-summable”.

Next we define some concepts which become important when the group under study is not sequentially Weil-complete.

Definition 2.2. We call a sequence \((g_n)_{n \in \mathbb{N}}\) of elements of \(G\)

• simply premultipliable if the sequence \(\prod_{k=1}^{n} g_k\) is a Cauchy sequence in \(G\).

• premultipliable in Bourbaki’s sense (B-premultipliable) if \((\prod_{n \in \Delta} g_n)_{\Delta \in \mathcal{F}(\mathbb{N})}\) is a Cauchy net in \(G\).

• super-premultipliable (or hereditarily premultipliable) in \(G\) if for every sequence \((m_k)_{k \in \mathbb{N}}\) of elements of \(\{0, 1\} \subset \mathbb{Z}\) the sequence \(\prod_{k=1}^{n} g^{m_k}\) is a Cauchy sequence in \(G\) (equivalently, for every strictly increasing sequence of natural numbers \((i_n)_{n \in \mathbb{N}}\) the sequence \((g_{i_n})_{n \in \mathbb{N}}\) is simply premultipliable).

• unconditionally super-premultipliable (or unconditionally hereditarily premultipliable) in \(G\) if for every sequence \((m_k)_{k \in \mathbb{N}}\) of elements of \(\{0, 1\} \subset \mathbb{Z}\) and for each bijective \(\sigma : \mathbb{N} \to \mathbb{N}\) the sequence \(\prod_{k=1}^{n} g^{m_{\sigma(k)}}\) is a Cauchy sequence in \(G\) (equivalently, if for every bijective \(\sigma : \mathbb{N} \to \mathbb{N}\) the sequence \((g_{\sigma(n)})_{n \in \mathbb{N}}\) is super-premultipliable).

• hyper-premultipliable in \(G\) if for every sequence \((m_k)_{k \in \mathbb{N}}\) of elements of \(\mathbb{Z}\) the sequence \(\prod_{k=1}^{n} g^{m_k}\) is a Cauchy sequence in \(G\).

• unconditionally hyper-premultipliable in \(G\) if for every sequence \((m_k)_{k \in \mathbb{N}}\) of elements of \(\mathbb{Z}\) and for each bijective \(\sigma : \mathbb{N} \to \mathbb{N}\) the sequence \(\prod_{k=1}^{n} g^{m_{\sigma(k)}}\) is a Cauchy sequence in \(G\) (equivalently, if for every bijective \(\sigma : \mathbb{N} \to \mathbb{N}\) the sequence \((g_{\sigma(n)})_{n \in \mathbb{N}}\) is hyper-premultipliable).
In case of abelian groups with the operation $+$, instead of the terms “simply premultipliable”, “$B$-premultipliable”, “(unconditionally) super-premultipliable”, “(unconditionally) hyper-premultipliable”, we shall use the terms: “simply presummable”, “$B$-presummable”, “(unconditionally) super-presummable”, “(unconditionally) hyper-presummable”.

It is clear that each of the multipliability properties listed above implies the corresponding premultipliability property. In the following diagrams we collect the remaining relations among these concepts. All the one-way implications are strict.

$$
\text{eventually neutral}
\Downarrow
\text{unc. hyper-multipl.} \implies \text{hyper-multipl.} \implies \text{hyper-convergent}
\Downarrow
\text{unc. super-multipl.} \implies \text{super-multipl.} \implies \text{$B$-multipl.}
\Downarrow
\text{simply multipl.}
$$

$$
\text{unc. hyper-premultipl.} \implies \text{hyper-premultipl.}
\Downarrow
\text{unc. super-premultipl.} \implies \text{super-premultipl.} \leftarrow \text{$B$-premultipl.}
\Downarrow
\text{simply premultipl.}
$$

Some of these implications are trivial; we next prove the remaining ones, as well as some other properties which are not included in the diagrams because they are true only in the abelian case.

The proof of the following lemma is standard:

**Lemma 2.3.** The sequence $(g_n)_{n \in \mathbb{N}}$ is $B$-premultipliable if and only if for every $U \in \mathcal{N}(G)$ there exists some $\Delta_U \in \mathcal{F}(\mathbb{N})$ such that for every $\Delta \in \mathcal{F}(\mathbb{N})$ with $\Delta \cap \Delta_U = \emptyset$ we have $\prod_{n \in \Delta} g_n \in U$. Moreover, $\Delta_U$ can be chosen of the form $\Delta_{n_U}$ for some $n_U \in \mathbb{N}$.
Proposition 2.4. A sequence \((g_n)_{n \in \mathbb{N}}\) is super-premultipliable if and only if it is B-premultipliable.

**Proof.** Suppose that \((g_n)_{n \in \mathbb{N}}\) is not B-premultipliable. By Lemma 2.3, there exist \(U \in \mathcal{N}(G)\) and a sequence \((\Delta_n)_{n \in \mathbb{N}}\) of finite subsets of \(\mathbb{N}\) such that \(\max \Delta_n < \min \Delta_{n+1}\) and \(\prod_{k \in \Delta_n} g_k \notin U\) for every \(n\). It is clear that the sequence \((\prod_{k=1}^{n} g_k^{m_k})_{n \in \mathbb{N}}\) is not a Cauchy sequence, if we let \((m_k)_{k \in \mathbb{N}}\) be the characteristic function of \(\bigcup_{n \in \mathbb{N}} \Delta_n\). Hence \((g_n)_{n \in \mathbb{N}}\) is not super-premultipliable.

The converse implication is easy to prove.

Proposition 2.5. Every super-multipliable sequence is B-multipliable.

**Proof.** Let \((g_n)_{n \in \mathbb{N}}\) be a super-multipliable sequence. By Proposition 2.4, it is B-premultipliable. Since the sequence \((\prod_{k \in \Delta_n} g_k)_{n \in \mathbb{N}}\) converges and the sequence \((\Delta_n)_{n \in \mathbb{N}}\) is cofinal for \(F(\mathbb{N})\), we get that the Cauchy net \((\prod_{k \in \Delta} g_k)_{\Delta \in F(\mathbb{N})}\) is convergent.

Proposition 2.6. Let \(G\) be an abelian group.

(a) If \((g_n)_{n \in \mathbb{N}}\) is a B-presummable sequence and \(\sigma : \mathbb{N} \to \mathbb{N}\) is bijective, then the sequence \((g_{\sigma(n)})_{n \in \mathbb{N}}\) is B-presummable as well.

(b) If \((g_n)_{n \in \mathbb{N}}\) is B-summable to \(g\) in \(G\) and \(\sigma : \mathbb{N} \to \mathbb{N}\) is bijective, then the sequence \((g_{\sigma(n)})_{n \in \mathbb{N}}\) is B-summable to \(g\) in \(G\) as well.

**Proof.** The proof is an easy consequence of the fact that \(\sum_{n \in \Delta} g_{\sigma(n)} = \sum_{j \in \sigma^{-1}(\Delta)} g_j\) for every \(\Delta \in F(\mathbb{N})\), which in turn follows from the commutativity of \(G\).

Lemma 2.7. Let \(G\) be a topological group and let \((g_n)_{n \in \mathbb{N}}\) be a sequence of elements of \(G\). If \((g_n)_{n \in \mathbb{N}}\) is hyper-convergent (resp. super-(pre)multipliable, hyper-(pre)multipliable) in \(G\) and \((i_n)_{n \in \mathbb{N}}\) is a strictly increasing sequence of natural numbers, then the sequence \((g_{i_n})_{n \in \mathbb{N}}\) is hyper-convergent (resp. super-(pre)multipliable, hyper-(pre)multipliable) in \(G\).

**Proof.** Suppose that \((g_n)_{n \in \mathbb{N}}\) is hyper-multipliable. Fix a sequence \(m : \mathbb{N} \to \mathbb{Z}\) and define \(\tilde{m} : \mathbb{N} \to \mathbb{Z}\) as follows: \(\tilde{m}_{i_n} = m_n, n = 1, 2, \ldots\) and \(\tilde{m}_j = 0, \forall j \in \mathbb{N} \setminus \{i_1, i_2, \ldots\}\). Then we have

\[
\prod_{k=1}^{i_n} g_k^{\tilde{m}_k} = \prod_{k=1}^{n} g_k^{m_k}, \; n = 1, 2, \ldots
\]
Since \((g_n)_{n \in \mathbb{N}}\) is hyper-multipliable in \(G\), the sequence \(\prod_{k=1}^{n} g_k^m\) converges to some \(g \in G\). From this and the above equality we get that the sequence \((\prod_{k=1}^{n} g_k^m)_{n \in \mathbb{N}}\) converges to \(g \in G\) as well. Since \(m : \mathbb{N} \to \mathbb{Z}\) is arbitrary, we have proved that the sequence \((g_n)_{n \in \mathbb{N}}\) is hyper-multipliable in \(G\). A similar argument can be used for the remaining types of sequences.

The following result has been proved in Lemma 4.3 of [4].

**Lemma 2.8.** Let \(G\) be a topological group and let \((g_n)_{n \in \mathbb{N}}\) be a sequence of elements of \(G\). If \((g_n)_{n \in \mathbb{N}}\) is unconditionally hyper-(pre)multipliable (resp. unconditionally super-(pre)multipliable) and \((i_n)_{n \in \mathbb{N}}\) is an injective sequence of natural numbers, then the sequence \((g_{i_n})_{n \in \mathbb{N}}\) is unconditionally hyper-(pre)multipliable (resp. unconditionally super-(pre)multipliable).

**Proof.** Suppose that \((g_n)\) is unconditionally hyper-multipliable. Let us see first that for every injective \(j : \mathbb{N} \to \mathbb{N}\), \((g_{j_n})_{n \in \mathbb{N}}\) is hyper-multipliable. Indeed, there exist a strictly increasing sequence \((l_n)_{n \in \mathbb{N}}\) in \(\mathbb{N}\) and a bijective \(s : \mathbb{N} \to \mathbb{N}\) with \(s(l_n) = j_n\) for every \(n\). The sequence \((g_{s(n)})_{n \in \mathbb{N}}\) is hyper-multipliable by definition. On the other hand, the sequence \(g_{j_n} = g_{s(l_n)}\) is one of its subsequences, hence (by Lemma 2.7) it is hyper-multipliable itself.

Now fix an injective \(i : \mathbb{N} \to \mathbb{N}\) and let us prove that \((g_{i_n})_{n \in \mathbb{N}}\) is actually unconditionally hyper-multipliable. Indeed, for any bijective \(\sigma : \mathbb{N} \to \mathbb{N}\) the map \(j = i \circ \sigma\) is injective and by the above argument, \((g_{j_n})_{n \in \mathbb{N}} = (g_{\sigma(j_n)})_{n \in \mathbb{N}}\) is hyper-multipliable.

The same general construction can be used for the remaining types of sequences.

**Lemma 2.9.** Let \(G\) be an abelian group and \((g_n)_{n \in \mathbb{N}}\) be a sequence of elements of \(G\). If \((g_n)_{n \in \mathbb{N}}\) is hyper-(pre)summable (resp. super-(pre)summable) and \((i_n)_{n \in \mathbb{N}}\) is an injective sequence of natural numbers, then the sequence \((g_{i_n})_{n \in \mathbb{N}}\) is hyper-(pre)summable (resp. super-(pre)summable).

In particular any hyper-(pre)summable (resp. super-(pre)summable) sequence is unconditionally hyper-(pre)summable (resp. unconditionally super-(pre)summable).

**Proof.** Suppose that \((g_n)_{n \in \mathbb{N}}\) is super-presummable. Let \((i_n)_{n \in \mathbb{N}}\) be an injective sequence of indices. Put \(i_n = \sigma(j_n)\) where \(\sigma : \mathbb{N} \to \mathbb{N}\) is bijective and \((j_n)_{n \in \mathbb{N}}\) is a strictly increasing sequence of indices. Using Propositions 2.4
and 2.6(a) we deduce that \((g_{\sigma(l)})_{l \in \mathbb{N}}\) is super-presummable. By Lemma 2.7, \((g_{\sigma(j_n)})_{n \in \mathbb{N}} = (g_{i_n})_{n \in \mathbb{N}}\) is super-presummable as well.

Now suppose that \((g_n)_{n \in \mathbb{N}}\) is super-summable. Clearly it suffices to show that \((g_{\sigma(j_n)})_{n \in \mathbb{N}}\) is simply summable for every injective sequence of indices \((i_n)_{n \in \mathbb{N}}\). Fix such a sequence \((i_n)_{n \in \mathbb{N}}\). Put \(i_n = j_{\sigma(n)}\) where \(\sigma : \mathbb{N} \to \mathbb{N}\) is bijective and \((j_k)_{k \in \mathbb{N}}\) is a strictly increasing sequence of indices. By Lemma 2.7, the sequence \((g_{j_n})_{n \in \mathbb{N}}\) is super-summable. By Proposition 2.5, this same sequence is B-summable. By Proposition 2.6(b), the sequence \((g_{j_{\sigma(n)}})_{n \in \mathbb{N}} = (g_{i_n})_{n \in \mathbb{N}}\) is B-summable as well. In particular it is simply summable.

The assertion concerning hyper-(pre)summability follows from the above proof and the fact that a sequence \((g_n)_{n \in \mathbb{N}}\) is hyper-(pre)summable if and only if for every sequence of integer numbers \((m_n)_{n \in \mathbb{N}}\) the sequence \((m_ng_n)_{n \in \mathbb{N}}\) is super-(pre)summable.

**Remark 2.10.** Rather unexpectedly, Lemma 2.9 may fail in the non-abelian case. Namely, in [4, Theorem 14.1] it is proved that in a (metrizable separable not Weil-complete) nonabelian topological group \(H\) there may exist a hyper-multipliable sequence \((b_n)_{n \in \mathbb{N}}\) and a bijection \(\varphi : \mathbb{N} \to \mathbb{N}\) such that the sequence \((b_{\varphi(n)})_{n \in \mathbb{N}}\) is not even simply multipliable in \(H\).

**Lemma 2.11.** Let \(G\) be a metrizable topological group, let \(d\) be a left-invariant metric which generates the topology of \(G\), and let \((g_n)_{n \in \mathbb{N}}\) be a sequence of elements of \(G\) such that

\[
\sum_{n=1}^{\infty} d(e, g_n) < \infty.
\]

Then the sequence \((g_n)_{n \in \mathbb{N}}\) is unconditionally super-premultipliable in \(G\).

**Proof.** Write \(b_n = \prod_{k=1}^{n} g_k, \; n = 1, 2, \ldots\) We have

\[
d(b_{n-1}, b_n) = d(b_{n-1}, b_{n-1}g_n) = d(e, g_n), \; n = 2, 3, \ldots
\]

since \(d\) is left-invariant. This and (1) imply

\[
\sum_{n=2}^{\infty} d(b_{n-1}, b_n) < \infty.
\]

We deduce that \((b_n)_{n \in \mathbb{N}}\) is a \(d\)-Cauchy sequence and thus (since \(d\) is left-invariant and generates the topology of \(G\)) a left Cauchy sequence in \(G\).
Now fix a sequence \((m_n)_{n \in \mathbb{N}}\) of elements of \(\{0, 1\} \subset \mathbb{Z}\) and an injective \(\sigma : \mathbb{N} \to \mathbb{N}\). It is well-known from analysis that
\[
\sum_{n=1}^{\infty} d(e, g_{\sigma(n)}^m) = \sum_{n=1}^{\infty} m_n d(e, g_{\sigma(n)}) \leq \sum_{n=1}^{\infty} d(e, g_n) < \infty.
\]
Hence we can apply the above argument to the sequence \((g_{\sigma(n)}^m)_{n \in \mathbb{N}}\) instead of \((g_n)_{n \in \mathbb{N}}\), and thus deduce that \((\prod_{k=1}^{n} g_{\sigma(k)}^m)_{n \in \mathbb{N}}\) is a Cauchy sequence in \(G\). Since \(\sigma\) and \((m_n)_{n \in \mathbb{N}}\) were arbitrary, we have proved that \((g_n)_{n \in \mathbb{N}}\) is unconditionally super-premultipliable in \(G\).

In the case of a metrizable group the hyper-convergent sequences can be characterized as follows.

**Lemma 2.12.** Let \(G\) be a metrizable topological group and \(d\) be a bounded metric which generates the topology of \(G\). Then for a sequence \((g_n)_{n \in \mathbb{N}}\) of elements of \(G\) the following conditions are equivalent:

1. \((g_n)_{n \in \mathbb{N}}\) is hyper-convergent in \(G\).
2. \(\lim_{n} \sup_{m \in \mathbb{Z}} d(e, g_n^m) = 0\).

**Proof.** (i)\(\Rightarrow\)(ii): Write \(\delta_n := \sup_{m \in \mathbb{Z}} d(e, g_n^m)\), \(n = 1, 2, \ldots\) Then we can find a sequence of integers \((m_n)_{n \in \mathbb{N}}\) such that
\[
\delta_n < d(e, g_n^{m_n}) + \frac{1}{n}, \quad n = 1, 2, \ldots
\]
Since \((g_n)_{n \in \mathbb{N}}\) is hyper-convergent in \(G\) and \(d\) generates the topology of \(G\), we have: \(\lim_n d(e, g_n^{m_n}) = 0\). This and the above inequality imply that \(\lim_n \delta_n = 0\) and (ii) is proved.

The implication (ii)\(\Rightarrow\)(i) is evident.

**Corollary 2.13.** Let \(G\) be a metrizable topological group, \(d\) a bounded metric which generates the topology of \(G\), and \((g_n)_{n \in \mathbb{N}}\) a hyper-convergent sequence in \(G\). Then \((g_n)_{n \in \mathbb{N}}\) has a subsequence \((g_{k_n})_{n \in \mathbb{N}}\) such that
\[
\sum_{n=1}^{\infty} \sup_{m \in \mathbb{Z}} d(e, g_{k_n}^m) < \infty.
\]
**Proposition 2.14.** Let $G$ be a metrizable topological group. $(g_n)_{n \in \mathbb{N}}$ be a hyper-convergent sequence in $G$. Then $(g_n)_{n \in \mathbb{N}}$ has a subsequence $(g_{k_n})_{n \in \mathbb{N}}$ which is unconditionally hyper-premultipliable in $G$.

**Proof.** Fix a bounded left-invariant metric $d$ which generates the topology of $G$. By Corollary 2.13, $(g_n)_{n \in \mathbb{N}}$ has a subsequence $(g_{k_n})_{n \in \mathbb{N}}$ such that

$$\sum_{n=1}^{\infty} \sup_{m \in \mathbb{Z}} d(e, g_{k_n}^m) < \infty.$$ 

This implies that, for any sequence of integers $(m_n)_{n \in \mathbb{N}},$

$$\sum_{n=1}^{\infty} d(e, g_{k_n}^{m_n}) < \infty.$$ 

Since $d$ is left-invariant, using Lemma 2.11 we conclude that $(g_{k_n}^{m_n})$ is unconditionally premultipliable in $G$. Since $(m_n)_{n \in \mathbb{N}}$ was arbitrary, this implies that $(g_{k_n})_{n \in \mathbb{N}}$ is unconditionally hyper-premultipliable in $G$.

**Corollary 2.15.** Let $G$ be a metrizable topological group.

(a) If $G$ contains an injective hyper-convergent sequence then $G$ contains an injective unconditionally hyper-premultipliable sequence.

(b) If $G$ contains an injective hyper-multipliable sequence, then $G$ contains an injective unconditionally hyper-premultipliable sequence.

**Proof.** (a) Let $(g_n)_{n \in \mathbb{N}}$ be an injective, hyper-convergent sequence in $G$. By Proposition 2.14, $(g_n)_{n \in \mathbb{N}}$ has a subsequence $(g_{k_n})_{n \in \mathbb{N}}$ which is unconditionally hyper-premultipliable in $G$. Clearly $(g_{k_n})_{n \in \mathbb{N}}$ is an injective sequence.

(b) follows from (a) because hyper-multipliable sequences are hyper-convergent.

**Remark 2.16.** Question 15.5 in [4] reads as follows: *Does there exist a (separable metrizable) group which contains an [injective] $f_\omega$-productive [=hyper-multipliable] sequence but does not contain any [injective] unconditionally $f_\omega$-Cauchy productive [=unconditionally hyper-premultipliable] sequence? Corollary 2.15(b) shows that the answer to this question is negative in the metrizable case. We do not know whether Corollary 2.15(b) remains true for non-metrizable groups.*
Example 2.17. A hyper-convergent sequence which is not simply premultipliable: Let $G = L_0([0,1])$ be the (complete separable metrizable not locally convex) space of all Lebesgue-measurable real functions on $[0,1]$ endowed with the topology of convergence in Lebesgue measure. It is known that the topology of $G$ is the one induced by the metric $d$ defined as follows:

$$d(x_1, x_2) = \int_0^1 \min\{|x_1(t) - x_2(t)|, 1\} \, dt, \quad x_1, x_2 \in G.$$  

Write $k_n = \frac{(n-1)n}{2}$ and $I_n = \{k \in \mathbb{N} : k_n < k \leq k_{n+1}\}$, for every $n \in \mathbb{N}$. Define a sequence $(g_k)_{k \in \mathbb{N}}$ as follows: for each $k \in \mathbb{N}$ find the unique $n \in \mathbb{N}$ such that $k \in I_n$ and let $g_k$ be the characteristic function of the interval $\left[\frac{k-k_n-1}{n}, \frac{k-k_n}{n}\right]$. For an arbitrary sequence $(m_k)_{k \in \mathbb{N}}$, we have $d(m_kg_k, 0) \leq \frac{1}{n}$ whenever $n \in \mathbb{N}$ and $k \in I_n$. This implies that $(g_k)_{k \in \mathbb{N}}$ is hyper-convergent. On the other hand, for every $n \in \mathbb{N}$ we have $d(\sum_{k=k_n+1}^{k_{n+1}} g_k, 0) = d(1,0) = 1$. Hence $(g_k)_{k \in \mathbb{N}}$ is not simply presummable.

Remark 2.18. In connection with Proposition 2.14 and Example 2.17, we note that in a sequentially complete abelian locally quasi-convex group $G$ any hyper-convergent sequence is unconditionally hyper-multipliable in $G$ (see [6]). Example 2.17 shows that a similar statement may not be true for a complete metrizable abelian non locally quasi-convex group $G$.

3. Absolutely productive sets. TAP, STAP and HTAP groups

The definition of an absolutely productive set can be formulated in terms of hyper-multipliable sequences, as follows.

Definition 3.1. (Cf. [8, Definition 4.1]) A subset $A$ of $G$ is absolutely productive in $G$ if every injective sequence $(g_n)_{n \in \mathbb{N}}$ of elements of $A$ is hyper-multipliable in $G$.

Note that, according to Definition 3.1, every finite subset $A$ of $G$ is absolutely productive in $G$.

Definition 3.2. ([8, Definition 4.5]) We say that $G$ is TAP if every absolutely productive set in $G$ is finite.

Proposition 3.3. A topological group $G$ is TAP if and only if every unconditionally hyper-multipliable sequence in $G$ is eventually neutral.
Proof. Suppose that $G$ is TAP. Let $(g_n)_{n \in \mathbb{N}}$ be an unconditionally hyper-multipliable sequence in $G$. It is clear that the set $A := \{g \in G : \exists n \in \mathbb{N} \text{ such that } g = g_n\}$ is absolutely productive, hence finite. Since $(g_n)_{n \in \mathbb{N}}$ converges to $e$, this sequence must be eventually neutral.

Now suppose that every unconditionally hyper-multipliable sequence in $G$ is eventually neutral. Let $A$ be an absolutely productive subset of $G$. Suppose that $A$ is infinite; pick an injective sequence $(g_n)_{n \in \mathbb{N}}$ in $A$. It is easy to see that $(g_n)_{n \in \mathbb{N}}$ is unconditionally hyper-multipliable, hence eventually neutral, a contradiction.

The following provides an example of a class of non-TAP groups.

Example 3.4. ([8, Lemma 4.4(i)]) An unconditionally hyper-multipliable sequence which is not eventually neutral: Put $G = \prod_{i \in \mathbb{N}} G_i$, where $(G_i)_{i \in \mathbb{N}}$ is a sequence of nontrivial Hausdorff topological groups. For each $i \in \mathbb{N}$ choose some $x_i \in G_i \setminus \{e_{G_i}\}$ and define a sequence $(g_n)_{n \in \mathbb{N}}$ as follows: for a fixed $n \in \mathbb{N}$, set $g_n(n) = x_n$ and $g_n(i) = e_{G_i}$ whenever $i \in \mathbb{N} \setminus \{n\}$. The sequence $(g_n)_{n \in \mathbb{N}}$ thus obtained is clearly unconditionally hyper-multipliable in $G$ but not eventually neutral. Hence $G$ is not TAP.

Motivated by the characterization of TAP groups given in Proposition 3.3, we give the following definitions.

Definition 3.5. We say that $G$ is

- HTAP (an abbreviation for “Hyper TAP”) if every hyper-multipliable sequence in $G$ is eventually neutral.

- STAP (an abbreviation for “Strictly TAP”) if every hyper-convergent sequence in $G$ is eventually neutral.

For a group $G$ to be TAP (resp. HTAP, STAP) it is sufficient that no injective sequence in $G$ is unconditionally hyper-multipliable (resp. hyper-multipliable, hyper-convergent) in $G$. This is an easy consequence of the fact that these properties are inherited by subsequences (Lemma 2.7).

Theorem 3.6. We have:

(a) $G \in \text{NSS} \Rightarrow G \in \text{STAP} \Rightarrow G \in \text{HTAP} \Rightarrow G \in \text{TAP}$.

(b) If $G$ is abelian then $G \in \text{HTAP} \Leftrightarrow G \in \text{TAP}$.
Proof. (a) Let us prove the first implication in (a). Fix a not eventually neutral sequence \((g_n)_{n \in \mathbb{N}}\) of elements of \(G\). Let us show that \((g_n)_{n \in \mathbb{N}}\) is not hyper-convergent. We can assume without loss of generality that \(g_n \neq e, n = 1, 2, \ldots\) Since \(G\) is NSS, we can find an open symmetric neighborhood \(U\) of \(e\) containing no nontrivial subgroups of \(G\). Fix \(n \in \mathbb{N}\). Since \(g_n \neq e\), the cyclic group \(\langle g_n \rangle\) is non-trivial, so \(\langle g_n \rangle \not\subset U\); hence there exists \(m_n \in \mathbb{Z}\) such that \(g_n^{m_n} \not\in U\). Consequently, we have constructed a sequence \((m_n)_{n \in \mathbb{N}}\) such that
\[
g_n^{m_n} \not\in U, \; n = 1, 2, \ldots
\]
This means that the sequence \((g_n)_{n \in \mathbb{N}}\) is not hyper-convergent to \(e\).

The remaining implications follow from the relations \([(g_n)_{n \in \mathbb{N}} \text{ unconditionally hyper-multipliable} \Rightarrow (g_n)_{n \in \mathbb{N}} \text{ hyper-multipliable} \Rightarrow (g_n)_{n \in \mathbb{N}} \text{ hyper-convergent}]\).

(b) follows from Lemma 2.9.

Remark 3.7. (a) The fact that \([G \in NSS \Rightarrow G \in TAP]\), which follows from Theorem 3.6(a), was proved as Theorem 4.9 in [8].

(b) Theorem 3.6(b) may fail in the nonabelian case. Namely, in [4, Theorem 14.1] the authors give an example of a (metrizable separable non Weil-complete) nonabelian TAP group which is not HTAP.

We shall see below that the properties TAP and STAP are equivalent for any metrizable Weil complete group (see Theorem 3.11).

Lemma 3.8. If \(G \notin NSS\) and \(d : G \times G \to \mathbb{R}_+\) is a continuous mapping with \(d(e,e) = 0\), then there exists a sequence \((g_n)_{n \in \mathbb{N}}\) of elements of \(G \setminus \{e\}\) such that
\[
\sum_{n=1}^{\infty} d(e, g_n^{m_n}) < \infty, \quad \forall (m_n)_{n \in \mathbb{N}} \in \mathbb{Z}^\mathbb{N}.
\] (2)

Proof. Let

\[
U_n = \{g \in G : d(e, g) < \frac{1}{2^n}\}, \; n = 1, 2, \ldots
\]

Since \(d\) is continuous on \(G \times G\), we have \(U_n \in \mathcal{N}(G), \; n = 1, 2, \ldots\) Since \(G \notin NSS\), there is a sequence \((H_n)_{n \in \mathbb{N}}\) of nontrivial subgroups of \(G\) such that
$H_n \subset U_n, \ n = 1, 2, \ldots$ Hence there exists a sequence $(g_n)_{n \in \mathbb{N}}$ of elements of $G$ such that

$$g_n \neq e, \ g_n \in H_n \subset U_n, \ n = 1, 2, \ldots$$  \hspace{1cm} (3)

Fix a sequence $(m_n)_{n \in \mathbb{N}}$ of integers. From (3), as $H_n, \ n = 1, 2, \ldots$ are subgroups of $G$, we get $g^{m_n}_n \in H_n \subset U_n, \ n = 1, 2, \ldots$ Hence,

$$d(e, g^{m_n}_n) < \frac{1}{2^n}, \ n = 1, 2, \ldots$$ \hspace{1cm} (4)

It follows from (4) that (2) is true.

**Theorem 3.9.** For a metrizable group $G$ the following are equivalent:

(i) $G \in \text{NSS}$.

(ii) $G \in \text{STAP}$.

**Proof.** (i) $\Rightarrow$ (ii) follows from Theorem 3.6(a).

(ii) $\Rightarrow$ (i). Suppose that $G \not\in \text{NSS}$. Fix any metric $d$ which generates the topology of $G$. By Lemma 3.8 there exists a non eventually neutral sequence $(g_n)_{n \in \mathbb{N}}$ of elements of $G$ which satisfies (2). It follows from (2) that $(g_n)_{n \in \mathbb{N}}$ hyper-converges to $e$. Consequently, $G \not\in \text{STAP}$.

The following proposition shows that the implication (ii) $\implies$ (i) in Theorem 3.9 may fail for a non-metrizable complete $\sigma$-compact topological abelian group.

**Proposition 3.10.** Let $E$ be an infinite-dimensional real Banach space, let $E^*$ be the space of all continuous linear functionals $x^* : E \to \mathbb{R}$, and let $G$ be the space $E^*$ endowed with the topology $\kappa$ of uniform convergence on compact subsets of $E$.

(a) $G$ is a non-metrizable complete $\sigma$-compact topological abelian group.

(b) $G \in \text{STAP}$.

(c) $G \in \text{NSS}$ iff $E$ is separable.

**Proof.** (a) is well-known.
(b) Let \((x^*_n)_{n \in \mathbb{N}}\) be a hyperconvergent sequence in \(G\); let us show that it is eventually neutral. Fix any sequence \((m_n)_{n \in \mathbb{N}}\) in \(\mathbb{Z}^\mathbb{N}\). By hypothesis the sequence \((nm_n x^*_n)_{n \in \mathbb{N}}\) converges to 0 in \(\kappa\); in particular (by Banach-Steinhaus theorem) it is bounded, i.e. there exists \(C > 0\) with \(\|nm_n x^*_n\| \leq C\) for every \(n \in \mathbb{N}\). From this we deduce \(m_n \|x^*_n\| \leq C/n\) for every \(n\). Since \((m_n)_{n \in \mathbb{N}}\) is arbitrary, the sequence \((\|x^*_n\|)_{n \in \mathbb{N}}\) is hyperconvergent in \(\mathbb{R}\), hence it is eventually neutral, and so is \((x^*_n)_{n \in \mathbb{N}}\).

(c) It is standard that the separability of \(E\) is equivalent to the fact that there exists a compact \(K \subset E\) which generates a dense subspace of \(E\). On the other hand, the subsets of the form

\[
K^o := \{x^* \in E^* : \sup_{y \in K} |x^*(y)| \leq 1\}
\]

(with \(K\) running over all compact subsets of \(E\)) constitute a basis of neighborhoods of zero in \(G\). Finally, by Hahn-Banach theorem, the neighborhood \(K^o\) contains only the trivial subgroup \(\{0\}\) of \(G\) if and only if the closed subspace of \(E\) generated by \(K\) coincides with \(E\).

**Theorem 3.11.** For a Weil-complete metrizable group \(G\) the following are equivalent:

(i) \(G \in \text{NSS}\).
(ii) \(G \in \text{STAP}\).
(iii) \(G \in \text{HTAP}\).
(iv) \(G \in \text{TAP}\).

**Proof.** The equivalence \((i) \iff (ii)\) follows from Theorem 3.9.

The implications \((ii) \Rightarrow (iii) \Rightarrow (iv)\) are evident.

\((iv) \Rightarrow (ii)\): Suppose that \(G \not\in \text{STAP}\). Then \(G\) contains an injective hyper-convergent sequence \((g_n)_{n \in \mathbb{N}}\). Since \(G\) is metrizable, according to Proposition 2.14, \((g_n)_{n \in \mathbb{N}}\) has an unconditionally hyper-premultipliable subsequence \((g_{k_n})_{n \in \mathbb{N}}\). Since \(G\) is Weil-complete, the sequence \((g_{k_n})_{n \in \mathbb{N}}\) is unconditionally hyper-multipliable in \(G\). Hence, \(G \not\in \text{TAP}\).
Remark 3.12. (1) In a preliminary version of the present paper the proof of the implication (iv) ⇒ (i) in Theorem 3.11 was incomplete. The argument therein proved only the weaker implication (iii) ⇒ (i). This gap was pointed out to the authors by Professor D. Dikranjan. A different proof of the implication (iv) ⇒ (i) is contained in the proof of [4, Corollary 5.6] (Corollary 4.6 in [3]).

(2) The implication (ii) ⇒ (i) in Theorem 3.11 may fail for a non-metrizable complete σ-compact topological abelian group (see Proposition 3.10).

(3) We do not know whether the implications (iv) ⇒ (i) or (iii) ⇒ (i) in Theorem 3.11 may fail for a metrizable Raikov-complete nonabelian group.

Since locally compact groups are Weil-complete, from Theorem 3.11 we get:

Corollary 3.13. For any locally compact metrizable group G, G ∈ NSS ⇔ G ∈ TAP.

Remark 3.14. Corollary 3.13 remains true without metrizability: it has been proved recently in [4, Theorem 10.13] (Theorem 10.8 in [3]) that a locally compact TAP group is NSS.

4. $C_p(X)$ and the STAP property

In this section $X$ will be a Tychonoff space and $C_p(X,G)$ will stand for the group of all continuous mappings $f : X → G$, endowed with the topology of pointwise convergence.

Theorem 4.1. [8, Theorem 5.3] A space $X$ is pseudocompact if and only if $C_p(X,\mathbb{R})$ has the TAP property.

To prove a STAP version of this result we need a simple lemma whose proof is omitted.

Lemma 4.2. Let $Y$ be an infinite topological space.

(a) If $Y$ is Hausdorff, then there exists $y \in Y$ and an injective sequence $(y_n)_{n \in \mathbb{N}}$ in $Y \setminus \{y\}$ which does not converge to $y$.

(b) If $Y$ is Hausdorff regular, then there exists a sequence $(V_n)_{n \in \mathbb{N}}$ of non-empty open subsets of $Y$ with pairwise disjoint closures.
Theorem 4.3. Let $X$ be a Tychonoff space. Then $C_p(X, \mathbb{R})$ has the STAP property if and only if $X$ is finite.

Proof. Suppose that $X$ is infinite and let us show that $C_p(X, \mathbb{R}) \not\in$ STAP. By Lemma 4.2(b) there exists a sequence $(V_n)_{n \in \mathbb{N}}$ of non-empty pairwise disjoint open subsets of $X$. Fix a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $X$ such that $x_n \in V_n$, $n = 1, 2, \ldots$ Using the complete regularity of $X$ we can find a sequence of continuous functions $f_n : X \to [0, 1]$, $n = 1, 2, \ldots$ such that

$$f_n(x_n) = 1; \quad f_n(x) = 0 \ \forall x \in X \setminus V_n, \quad (n = 1, 2, \ldots)$$

From (5) it follows in particular that $f_n \neq 0$, $n = 1, 2, \ldots$, where 0 stands for the identically zero function, the neutral element of $C_p(X, \mathbb{R})$. Consequently, $(f_n)_{n \in \mathbb{N}}$ is not an eventually neutral sequence in $C_p(X, \mathbb{R})$. Now we will show that $(f_n)_{n \in \mathbb{N}}$ is a hyper-convergent sequence in $C_p(X, \mathbb{R})$ and thus $C_p(X, \mathbb{R}) \not\in$ STAP.

Fix $x \in X$. Let us prove that $(f_n(x))_{n \in \mathbb{N}}$ is an eventually neutral sequence in $\mathbb{R}$.

If $x \in X \setminus \bigcup_{n \in \mathbb{N}} V_n$, then we deduce from (5) that $f_n(x) = 0$, $n = 1, 2, \ldots$

If $x \in \bigcup_{n \in \mathbb{N}} V_n$, then $x \in V_{n_x}$ for some $n_x \in \mathbb{N}$. Since the sets $V_n$, $n = 1, 2, \ldots$ are pairwise disjoint, $x \not\in V_n$ for every $n \neq n_x$ and in particular, using (5) once again,

$$n \in \mathbb{N}, \quad n > n_x \Rightarrow f_n(x) = 0.$$ 

Hence $(f_n(x))_{n \in \mathbb{N}}$ is an eventually neutral sequence in $\mathbb{R}$.

This clearly implies that for every sequence $(m_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{Z}$ the sequence $(m_n f_n(x))_{n \in \mathbb{N}}$ is again an eventually neutral sequence in $\mathbb{R}$. In particular, we have that $\lim_n m_n f_n(x) = 0$ for every sequence $(m_n)_{n \in \mathbb{N}}$ in $\mathbb{Z}$ and this, by the definition of the topology of $C_p(X, \mathbb{R})$, means that $(f_n)_{n \in \mathbb{N}}$ is a hyper-convergent sequence in $C_p(X, \mathbb{R})$.

We have proved that if $C_p(X, \mathbb{R})$ has the STAP property, then $X$ is finite. Conversely, if $X$ is finite, then clearly $C_p(X, \mathbb{R}) = \mathbb{R}^X \in$ STAP.

5. NSS topological vector groups

In this introductory section we consider the NSS property for group-topologized real vector spaces.

A nonempty subset $A$ of a real vector space $E$ is called balanced if $tA := \{ta : a \in A\}$ is a subset of $A$ for every $t \in \mathbb{R}$, $|t| \leq 1$. If $A$ is balanced, then
0 ∈ A and A is symmetric. If 0 ∈ A and A is symmetric and convex, then A is balanced (the converse is not true when dim(E) > 1).

**Lemma 5.1.** Let A be a balanced subset of a real vector space E and a ∈ A be such that ⟨a⟩ ⊂ A. Then RA ⊂ A.

**Proof.** Fix t ∈ R. Find a natural number n such that |t/n| ≤ 1. Then na ∈ ⟨a⟩ ⊂ A. Hence, as na ∈ A and A is balanced, we get: ta = (t/n)(na) ∈ A.

The following is well-known.

**Lemma 5.2.** Let E be a topological vector space over R. Then

(a) For every $t \in R \setminus \{0\}$ the map $x \mapsto tx$ is a linear homeomorphism of E onto E.

(b) The balanced members of $N(E)$ form a basis of neighborhoods of zero in E.

We will call group-topologized vector space a vector space E endowed with a topology τ such that $(E, τ)$ is a topological Abelian group with respect to addition.

A group-topologized vector space G over R is said to be

- a topological vector group over R if for every $t \in R$ the map $x \mapsto tx$ is continuous.
- locally balanced if the balanced members of $N(E)$ form a basis of $N(E)$.
- locally convex if the convex symmetric members of $N(E)$ form a basis of neighborhoods of zero in E.

**Lemma 5.3.** Let G be a group-topologized vector space over R. Then

(a) (cf. [5, Proposition 2.3]) If G is locally balanced, then G is a topological vector group over R.

(b) If G is locally convex, then G is a topological vector group over R.
Proof. (a) Fix \( t \in \mathbb{R} \) and \( U \in \mathcal{N}(E) \). If \( t \in [-1, 1] \), find a balanced \( V \in \mathcal{N}(E) \) with \( V \subset U \). Then \( tV \subset V \subset U \). Hence the map \( x \mapsto tx \) is continuous at \( 0 \in E \) and hence, is continuous. If \( |t| > 1 \), we can write \( t = s + m \) for some \( s \in [-1, 1] \) and \( m \in \mathbb{Z} \). Then, as we have seen, the map \( x \mapsto sx \) is continuous; since \( G \) is a topological group, the map \( x \mapsto mx \) is continuous as well. Consequently the map \( x \mapsto tx \), as the pointwise sum of two continuous mappings, is continuous as well.

(b) follows from (a).

Lemma 5.4. Let \( G \) be a group-topologized vector space over \( \mathbb{R} \). Consider the family \( \{ h_t : t \in [0, 1] \} \) of group endomorphisms of \( G \), where \( h_t(x) := tx \) for every \( t \in [0, 1] \) and every \( x \in G \). The following are equivalent:

(i) \( G \) is a locally balanced topological vector group over \( \mathbb{R} \).

(ii) \( \{ h_t : t \in [0, 1] \} \) is an equicontinuous family.

Proof. (i) \( \Rightarrow \) (ii): It is sufficient to show that \( \{ h_t : t \in [0, 1] \} \) is equicontinuous at \( 0 \). So, fix \( U \in \mathcal{N}(E) \) and take a balanced set \( V \in \mathcal{N}(E) \) such that \( V \subset U \). Then \( h_t(V) \subset U \) for each \( t \in [0, 1] \).

(ii) \( \Rightarrow \) (i): By Lemma 5.3(a) it is sufficient to show that \( G \) is locally balanced. So fix \( U \in \mathcal{N}(E) \) and let us find a balanced \( V \in \mathcal{N}(E) \) such that \( V \subset U \). Since \( \{ h_t : t \in [0, 1] \} \) is equicontinuous at \( 0 \), there is \( W \in \mathcal{N}(E) \) such that \( h_t(W) = tW \subset U \forall t \in [0, 1] \). Write \( V := \bigcup_{t \in [0, 1]} tW \). Then \( V \) is balanced, \( V \in \mathcal{N}(E) \) because \( V \supset W \), and \( V \subset U \).

Lemma 5.5. For a group-topologized vector space \( G \) over \( \mathbb{R} \) the following are equivalent:

(i) \( G \) is a metrizable locally balanced topological vector group over \( \mathbb{R} \).

(ii) The topology of \( G \) can be induced by a translation-invariant metric \( d \) which has the following additional property:

\[
g \in G, \alpha_1, \alpha_2 \in \mathbb{R}, |\alpha_1| \leq |\alpha_2| \Rightarrow d(\alpha_1 \cdot g, 0) \leq d(\alpha_2 \cdot g, 0) \quad (6)
\]
Proof. \((i) \Rightarrow (ii)\): Fix a bounded translation-invariant metric \(\rho\) for \(G\) and define a mapping \(d : G \times G \to \mathbb{R}_+\) by the equality:

\[
d(x, y) = \sup_{\alpha \in [-1, 1]} \rho(\alpha x, \alpha y) \quad \forall x, y \in G.
\]

Then \(d \geq \rho\) and it is easy to verify that \(d\) is a translation-invariant metric with property (6). It remains to show that the topology in \(G\) induced by \(d\) is coarser than the original topology of \(G\) (induced by \(\rho\)). So, let us take \(\varepsilon > 0\) and find \(V \in \mathcal{N}(G)\) such that \(V \subset \{x \in G : d(0, x) < \varepsilon\}\). Since \(\{x \in G : \rho(0, x) < \varepsilon/2\} \in \mathcal{N}(G)\) and \(G\) is locally balanced, we can find a balanced \(V \in \mathcal{N}(G)\) such that \(V \subset \{x \in G : \rho(0, x) < \varepsilon/2\}\). Since \(V\) is balanced, the last inclusion implies that \(\rho(0, \alpha x) < \varepsilon/2\) \(\forall \alpha \in [-1, 1], \forall x \in V\). Hence, \(d(0, x) \leq \varepsilon/2 < \varepsilon\) \(\forall x \in V\), i.e., \(V \subset \{x \in G : d(0, x) < \varepsilon\}\) and the implication is proved.

\((ii) \Rightarrow (i)\): Clearly \((ii)\) implies that \(G\) is metrizable and locally balanced. Then by Lemma 5.3\((a)\), \(G\) is a topological vector group.

We need the following easy fact.

Proposition 5.6. For a locally balanced Hausdorff topological vector group \(G\) over \(\mathbb{R}\), the following are equivalent:

\((i)\) \(G \notin \text{NSS}\).

\((ii)\) Every \(U \in \mathcal{N}(G)\) contains a 1-dimensional vector subspace of \(G\).

Proof. \((i) \Rightarrow (ii)\): Fix \(U \in \mathcal{N}(G)\) and find a balanced \(V \in \mathcal{N}(E)\) with \(V \subset U\). By \((i)\) there is a nontrivial subgroup \(H\) of \(G\) such that \(H \subset V\). Pick some \(x \in H \setminus \{0\}\). Then \(\langle x \rangle \subset H \subset V\). From this and Lemma 5.1 we deduce \(\mathbb{R}x \subset V \subset U\).

\((ii) \Rightarrow (i)\) is true because a 1-dimensional vector subspace of \(G\) is also a nontrivial subgroup of \(G\).

It is clear that if \(X\) is an infinite set, then \(\mathbb{R}^X \notin \text{NSS}\). This also follows from the following statement, which is not evident at once.

Proposition 5.7. Let \(X\) be a nonempty set and \(E\) be a vector subspace of \(\mathbb{R}^X\) endowed with the induced topology. If \(E \in \text{NSS}\), then \(\dim(E) < \infty\).
Proof. For a nonempty subset $\Delta \subset X$ and $\varepsilon > 0$, put

$$V_{\Delta, \varepsilon} := \{ f \in E : f(\Delta) \subset [-\varepsilon, \varepsilon] \}.$$ 

The sets $V_{\Delta, \varepsilon}$ form a basis for $\mathcal{N}(E)$ as $\Delta$ runs over all finite subsets of $X$ and $\varepsilon$ runs over all positive real numbers.

As $E \in \text{NSS}$, by Proposition 5.6 there exist a finite nonempty $\Delta \subset X$ and $\varepsilon > 0$ such that $V_{\Delta, \varepsilon}$ does not contain any 1-dimensional vector subspace of $E$. Consider now the mapping $u : E \to \mathbb{R}^\Delta$ defined by the rule: $u(f) = f |_\Delta$, where $f \in E$. Clearly $u$ is linear. Let us see that $u$ is injective too. Take $f \in E$ with $u(f) = 0 \in \mathbb{R}^\Delta$. This means that $f(\Delta) = \{0\}$. Then $tf(\Delta) = \{0\}, \forall t \in \mathbb{R}$ and in particular, $tf \in V_{\Delta, \varepsilon} \forall t \in \mathbb{R}$. From this, since $V_{\Delta, \varepsilon}$ does not contain any 1-dimensional vector subspace of $E$, we get that $f = 0$. Therefore $u$ is injective and hence, it is a vector space isomorphism between $E$ and $u(E) \subset \mathbb{R}^\Delta$. Consequently, $\dim(E) = \dim(u(E)) \leq \dim(\mathbb{R}^\Delta) = \text{Card}(\Delta) < \infty$.

Remark 5.8. With the notations of Proposition 5.7, the stronger implication $[E \in \text{STAP} \Rightarrow \dim(E) < \infty]$ is not true in general. Indeed, let $E$ be an infinite-dimensional real Banach space, let $E^*$ be the space of all continuous linear functionals $x^* : E \to \mathbb{R}$, and let $G$ be the space $E^*$ endowed with the topology of pointwise convergence on points of $E$. Then $G$ is a subspace of $\mathbb{R}^E$, $E$ is STAP (the same proof as in Proposition 3.10), but $G$ is infinite-dimensional (this is a consequence of the Hahn-Banach theorem).

The next section is dedicated to the group analogs of the following result.

Theorem 5.9. ([1, Theorem 9 and Corollary]) For a complete metrizable topological vector space $E$ over $\mathbb{R}$ the following are equivalent:

(i) $E$ contains a vector subspace topologically isomorphic to $\mathbb{R}^\mathbb{N}$.

(ii) For every $U \in \mathcal{N}(E)$, there exists an element $x \in E \setminus \{0\}$ such that for every real $t$ the relation $tx \in U$ holds [i.e. $E \not\in \text{NSS}$, see Proposition 5.6].

(iii) There exists a not eventually neutral sequence $(x_n)$ in $E$ such that for every real sequence $(t_n)$ the series $\sum_{n=1}^{\infty} t_n x_n$ converges.
6. Locally root invariant, NSS groups

Next we will define an analog of local balancedness which makes sense for general topological groups.

Let $G$ be an abelian group and $x$ an element of $G$. Let $D(x, G)$ be the set
\{ $y \in G : ny = x$ for some $n \in \mathbb{N}$ \}. (Note that $D(0, G)$ is the torsion subgroup of $G$.) We say that a subset $U$ of $G$ is root invariant in $G$ if $D(x, G) \subset U$ for every $x \in U$.

**Definition 6.1.** Let $G$ be a topological abelian group. We say that $G$ is locally root invariant if the root invariant members of $\mathcal{N}(G)$ form a basis of neighborhoods of zero in $G$.

It is clear that every locally root invariant Hausdorff topological abelian group is torsion-free, and that every topological subgroup of a locally root invariant topological group is locally root invariant itself.

Let $G$ be an abelian group and $x \in G$ be a uniquely divisible element. For a fixed $n \in \mathbb{N}$ we denote by $\frac{1}{n}x$ the unique element $y \in G$ for which $ny = x$. The following assertion is a group version of Lemma 5.4:

**Lemma 6.2.** Let $G$ be a Hausdorff topological abelian divisible group. Consider the family \{ $h_{1/n} : n \in \mathbb{N}$ \} of group endomorphisms of $G$, where $h_{1/n}(x) := \frac{1}{n}x$ for every $n \in \mathbb{N}$ and every $x \in G$. The following are equivalent:

(i) $G$ is locally root invariant.

(ii) $G$ is torsion-free, uniquely divisible and the family \{ $h_{1/n} : n \in \mathbb{N}$ \} is equicontinuous.

**Proof.** 
(i) $\Rightarrow$ (ii): As $G$ is a locally root invariant Hausdorff topological abelian group, it is torsion-free. Clearly a torsion-free divisible abelian group is uniquely divisible. Since $G$ is abelian, it follows also that for a fixed $n \in \mathbb{N}$ the mapping $x \mapsto \frac{1}{n}x$ is a group homomorphism. So, it is sufficient to show that the sequence $h_{1/n}$, $n \in \mathbb{N}$ is equicontinuous at 0.

Fix $U \in \mathcal{N}(E)$. Find a root-invariant $V \in \mathcal{N}(E)$ with $V \subset U$. It is clear that \{ $\frac{1}{n}x : x \in V$ \} $\subset V \subset U$ and hence, $h_{1/n}(V) \subset U$ for every $n \in \mathbb{N}$.
(ii) ⇒ (i): Fix $U \in \mathcal{N}(E)$ and let us find a root-invariant $V \in \mathcal{N}(E)$ such that $V \subset U$. Since $\{h_{1/n} : n \in \mathbb{N}\}$ is equicontinuous at 0, there is $W \in \mathcal{N}(E)$ such that $h_{1/n}(W) = \frac{1}{n}W \subset U \ \forall n \in \mathbb{N}$. Put $V := \bigcup_{n \in \mathbb{N}} h_{1/n}(W)$. Then $V \subset U$, and $V \in \mathcal{N}(E)$ because $V \supset W$.

Moreover, $V$ is root-variant. Indeed, fix $x \in V$; let us see that $D(x,G) \subset V$. Since $x \in V$, we have $x \in h_{1/n}(W)$ for some $n \in \mathbb{N}$. If $y \in D(x,G)$, then $my = x$ for some $m \in \mathbb{N}$. Since $G$ is uniquely divisible, $y = h_{1/m}(x) \in h_{1/m}(h_{1/n}(W)) \subset h_{1/mn}(W)$ and therefore $y \in V$.

We also have the following analog of Lemma 5.5:

**Lemma 6.3.** For a topological abelian divisible group $G$ the following are equivalent:

(i) $G$ is a metrizable, locally root invariant topological group.

(ii) The topology of $G$ can be induced by a translation-invariant metric $d$ which has the following additional property:

$$g \in G, n \in \mathbb{N} \Rightarrow d(g,0) \leq d(n,0).$$

(7)

The implication (ii) ⇒ (i) is true for a not necessarily divisible group $G$.

**Proof.** (i) ⇒ (ii): By Lemma 6.2, $G$ is uniquely divisible. Fix a bounded translation-invariant metric $\rho$ on $G$ which generates the topology of $G$, and define a mapping $d : G \times G \to \mathbb{R}_+$ as follows:

$$d(x, y) = \sup_{n \in \mathbb{N}} \rho\left(\frac{1}{n}x, \frac{1}{n}y\right) \ \forall x, y \in G.$$ 

Then $d \geq \rho$ and it is easy to verify that $d$ is a translation-invariant metric with property (7). It remains to show that the topology in $G$ induced by $d$ is coarser than the original topology of $G$ (induced by $\rho$). Fix an $\varepsilon > 0$. We must find $V \in \mathcal{N}(G)$ such that $V \subset \{x \in G : d(0, x) < \varepsilon\}$. To this end, note that since $\{x \in G : \rho(0, x) < \varepsilon/2\} \in \mathcal{N}(G)$ and $G$ is locally root invariant, we can find a root invariant $V \in \mathcal{N}(G)$ such that $V \subset \{x \in G : \rho(0, x) < \varepsilon/2\}$. Since $V$ is root invariant, we conclude that $x \in V, a \in D(x,G) \Rightarrow \rho(0, a) < \varepsilon/2$.

Hence, for all $x \in V$ we have $d(0, x) \leq \varepsilon/2 < \varepsilon$. This shows that $d$ induces on $G$ its original topology.
\( (ii) \Rightarrow (i) \) is clear.

Let us denote by MMP the class of all locally root invariant topological abelian groups \( G \) which are metrizable by a translation-invariant metric \( d \) with property (7). This class is rather wide, as we can deduce from Lemmas 5.5 and 6.3 and the fact that it is invariant under forming subgroups. However, we do not know if it contains all locally root invariant, metrizable abelian groups.

**Lemma 6.4.** Let \( G \) be a metrizable topological abelian non-NSS group. Let \( d \) be a bounded translation-invariant metric which generates the topology of \( G \). Then there are sequences \( (\delta_n)_{n \in \mathbb{N}} \) in \( (0,1) \) and \( (g_n)_{n \in \mathbb{N}} \) in \( G \setminus \{0\} \) with \( \delta_n = \sup_{m \in \mathbb{Z}} d(mg_n,0) \) such that \( \delta_{n+k} < \frac{1}{4^n} \delta_n \) for all \( n, k \in \mathbb{N} \). In particular, \( \sum_{i=n+1}^{\infty} \delta_i < \frac{1}{3} \delta_n \) for every \( n \).

**Proof.** Since \( G \) is not NSS, by Theorem 3.9 it is not STAP either. Let \( (x_k)_{k \in \mathbb{N}} \) be a hyper-convergent sequence in \( G \) such that \( x_k \neq 0 \) for every \( k \in \mathbb{N} \). By Lemma 2.12, we have that \( \lim_k \sup_{m \in \mathbb{Z}} d(mx_k,0) = 0 \). Let \( (k_n)_{n \in \mathbb{N}} \) be a strictly increasing sequence of integers such that the sequence \( \delta_n := \sup_{m \in \mathbb{Z}} d(mg_{k_n},0) \) satisfies \( 0 < \delta_1 < 1, 0 < \delta_{n+1} < \frac{1}{4^n} \delta_n \) for every \( n \). Put \( g_n = x_{k_n} \) for every \( n \in \mathbb{N} \). The sequences \( (\delta_n) \) and \( (g_n) \) satisfy the required properties.

For any sequence \( g := (g_n)_{n \in \mathbb{N}} \) of elements of a topological abelian group \( G \) we write

\[
E_g := \{ m \in \mathbb{Z}^\mathbb{N} : (\sum_{k=1}^{n} m(k)g_k)_{n \in \mathbb{N}} \text{ converges in } G \}.
\]

Define a mapping \( u_g : E_g \to G \) by the equality

\[
u_g(m) = \sum_{k=1}^{\infty} m(k)g_k := \lim_n \sum_{k=1}^{n} m(k)g_k, \quad m \in E_g.
\]

Observe that for every \( g := (g_n)_{n \in \mathbb{N}} \) we have:

- \( \mathbb{Z}^{(\mathbb{N})} \subset E_g \);
- \( E_g \) is a subgroup of \( \mathbb{Z}^{\mathbb{N}} \),
\* \( u_g : E_g \to G \) is a group homomorphism.

**Lemma 6.5.** With the above notations, if \( G \) is complete metrizable and non-NSS and \((g_n)\) is as in Lemma 6.4, then \( E_g = \mathbb{Z}^N \).

**Proof.** Let \( d \) be a bounded, translation-invariant metric which generates the topology of \( G \). Fix any \( m = (m(n))_{n \in \mathbb{N}} \in \mathbb{Z}^N \). Then

\[
\sum_{n=2}^{\infty} d(m(n)g_n, 0) \leq \sum_{n=2}^{\infty} \delta_n \leq \frac{1}{3} \delta_1 < \infty.
\]

This clearly implies that the sequence \( \sum_{k=1}^{n} m(k)g_k \) is a Cauchy sequence, hence it converges in \( G \).

**Lemma 6.6.** With the above notation, if \( G \) is a metrizable topological abelian non-NSS group and \((\delta_n)\) and \( g = (g_n) \) are as in Lemma 6.4, then the homomorphism \( u_g \) is continuous.

**Proof.** Fix a sequence \((m_n)_{n \in \mathbb{N}}\) in \( E_g \) which converges pointwise to zero. Fix any \( \varepsilon > 0 \). Find \( k_0 \in \mathbb{N} \) such that \( \delta_{k_0} \leq 3\varepsilon \) and \( n \in \mathbb{N} \) such that \( m_n(k) = 0 \) for every \( k \leq k_0 \). Then for every \( n' \geq n \) we get

\[
d(u_g((m_{n'})), 0) = d\left( \sum_{k=1}^{\infty} m_{n'}(k)g_k, 0 \right) = d\left( \sum_{k=1}^{\infty} m_{n'}(k)g_k, 0 \right) \leq \sum_{k=k_0+1}^{\infty} \delta_k < \frac{1}{3} \delta_{k_0} \leq \varepsilon
\]

so \( u_g((m_n)) \to 0 \).

**Theorem 6.7.** Let \( G \) be a non-NSS group which is in MMP. Let \( d \) be a bounded, translation-invariant metric which generates the topology of \( G \) and satisfies property (7). If \( g = (g_n) \) is as in Lemma 6.4, then \( u_g : E_g \to G \) is a topological embedding.

**Proof.** By Lemma 6.6, \( u_g \) is continuous. Let us see that given any sequence \((m_n) \in E_g\), if we have \( \lim_n u_g(m_n) = 0 \), then \( \{m_n(k) : n \in \mathbb{N}\} \) is eventually zero for every \( k \) (that is, \((m_n)\) converges pointwise to zero). This clearly will imply that \( u_g \) is both injective and open onto its image.

Indeed, suppose that \( \lim_n \sum_{k=1}^{\infty} m_n(k)g_k = 0 \) but for some \( k \in \mathbb{N} \) there are infinitely many nonzero \( m_n(k) \). Let \( k_1 \) be the smallest index for which this is true. Let \((n_j)_{j \in \mathbb{N}}\) be a strictly increasing sequence of indices with
We may suppose that $m_{n_j}(k) = 0$ for every $j \in \mathbb{N}$ and every $k < k_1$. Hence we get, for every $j$,

$$
\sum_{k=1}^{\infty} m_{n_j}(k)g_k = m_{n_j}(k_1)g_{k_1} + \sum_{k=k_1+1}^{\infty} m_{n_j}(k)g_k
$$

We now multiply the above equality by an arbitrary $l \in \mathbb{Z}$:

$$
lm_{n_j}(k_1)g_{k_1} = l\sum_{k=1}^{\infty} m_{n_j}(k)g_k - \sum_{k=k_1+1}^{\infty} lm_{n_j}(k)g_k \quad (l \in \mathbb{Z}, j \in \mathbb{N})
$$

By the triangle inequality

$$
d(lm_{n_j}(k_1)g_{k_1}, 0) \leq ld(\sum_{k=1}^{\infty} m_{n_j}(k)g_k, 0) + \sum_{k=k_1+1}^{\infty} d(lm_{n_j}(k)g_k, 0)
$$

$$
\leq ld(\sum_{k=1}^{\infty} m_{n_j}(k)g_k, 0) + \frac{1}{3}\delta_{k_1} \quad (l \in \mathbb{Z}, j \in \mathbb{N})
$$

By our choice of $d$ and the fact that $m_{n_j}(k_1) \neq 0$, we have $d(lm_{n_j}(k_1)g_{k_1}, 0) \geq d(lg_{k_1}, 0)$ for all $l$ and $j$. Since the sequence $(\sum_{k=1}^{\infty} m_{n_j}(k)g_k)_{j \in \mathbb{N}}$ converges to zero, we deduce that

$$
d(lg_{k_1}, 0) \leq \frac{1}{3}\delta_{k_1} \quad (l \in \mathbb{Z})
$$

which is a contradiction with the definition of $\delta_{k_1}$.

**Remark 6.8.** The idea of the proof of Lemma 6.4 is taken from [7, Theorem 1], while the argument of the proof of Theorem 6.7 is modelled on the proof of [1, Theorem 9].

**Theorem 6.9.** For a group $G \in \text{MMP}$ the following are equivalent:

(i) $G \in \text{NSS}$.

(ii) $G \in \text{STAP}$.

(iii) $G$ does not contain any subgroup topologically isomorphic to $\mathbb{Z}^{(N)}$.

**Proof.** (i) $\Leftrightarrow$ (ii) follows from Theorem 3.9.
(ii) ⇒ (iii) is clear.

(iii) ⇒ (i) follows from Theorem 6.7.

**Theorem 6.10.** For a complete group $G \in \text{MMP}$ the following are equivalent:

(i) $G \in \text{NSS}$.

(ii) $G \in \text{STAP}$.

(iii) $G \in \text{TAP}$.

(iv) $G$ does not contain any subgroup topologically isomorphic to $\mathbb{Z}^\mathbb{N}$.

(v) $G$ does not contain any subgroup topologically isomorphic to $\mathbb{Z}^\mathbb{N}$.

**Proof.** The equivalences (i) ⇔ (ii) ⇔ (iv) were proved in Theorem 6.9.

(ii) ⇒ (iii) was proved in Theorem 3.11.

(iv) ⇒ (v) is trivial.

(v) ⇒ (i) follows from Lemma 6.5 and Theorem 6.7.

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