Schwartz groups and convergence of characters

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Abstract

It is proved that a locally quasi-convex group is a Schwartz group if and only if every continuously convergent filter on its dual group converges locally uniformly. We also show that for metrizable separable groups a similar result remains true when filters are replaced by sequences. As an ingredient in the proofs of these results, we obtain a Schauder-type theorem on compact homomorphisms acting between the natural group analogues of normed spaces.

Keywords. Schwartz group; Schwartz space; locally uniform convergence; continuous convergence.

MSC codes. 22A05, 54A20, 46A11.

1 Introduction

The notion of a Schwartz locally convex space was introduced by A. Grothendieck in [18, Definition 5]. Almost all results about Schwartz spaces obtained in [18] were also included in his monograph [19]. Although nuclear locally convex spaces, which form a proper subclass of that of Schwartz spaces, had been defined earlier by Grothendieck himself in [16, 17], such spaces are not mentioned in [18, 19] at all. Later S. Rolewicz defined similar concepts in the class of, not necessarily locally convex, metrizable topological vector spaces ([29, 30]).

A definition of nuclear topological abelian groups and the study of the class thus obtained were developed by W. Banaszczyk in [4]. His work led to proving the validity of such important results as the Pontryagin-van Kampen duality theorem in its strong form for the class of Čech-complete nuclear groups (see [1, Th. 20.40]). To this day, no examples of strongly Pontryagin reflexive groups have been found outside this class.

Some years later, a notion of Schwartz topological abelian group appeared in [2]. In relation to the present paper's subject, it was proved there that every nuclear group is a locally quasi-convex Schwartz group and that the given definition is coherent with that of a Schwartz topological vector space. Later L. Außenhofer proved in [3] that locally quasi-convex Schwartz groups satisfy the Glicksberg property (a similar statement for nuclear groups was obtained earlier by W. Banaszczyk and E. Martín in [5]).

In this note we are interested in showing how the property of being a Schwartz group is reflected in, or can be deduced from, convenient features of the dual group. Our motivation for this comes from [18, Proposition 17] where, besides the internal definition of a Schwartz space, the following dual characterization is given: An (F)-space E is Schwartz iff it is a separable (M)-space, and every strongly convergent sequence in the dual E' converges uniformly on some neighborhood of the origin of E. Later several papers dealt with similar characterizations (see [22, 26]); in [15, Korollar 1, p. 178] (see also [23, Theorem 11.6.3]) the following improvement of Grothendieck's result can be found: a separable (F)-space E is Schwartz iff every pointwise convergent sequence in the dual E', converges uniformly on some neighborhood of the origin of E. Finally, in [8, 9, 27] the analogous statement was obtained without the separability assumption. In [23, p. 201] a Hausdorff locally convex space E *is defined* to be a Schwartz space if every continuously convergent filter on E' converges equicontinuously (i. e. uniformly on some neighborhood of the origin in E), and in [23, Theorem 10.4.1] this new definition is shown to be equivalent with Grothendieck's original one. This characterization of Schwartz spaces is presented also in [6, Corollary 4.3.43].

In the present article we prove the statements formulated in the Abstract, and thus obtain characterizations of locally quasi-convex Schwartz groups in the spirit of Grothendieck-Floret-Jarchow's results.

2 First definitions and results

All groups under consideration will be abelian. We do not assume topological groups to be Hausdorff unless explicitly stated. The set of neighborhoods of the neutral element in the group G will be denoted by $\mathcal{N}(G)$.

Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle group. We write $\mathbb{T}_+ = \{z \in \mathbb{T} : \operatorname{Re} z \geq 0\}$. In what follows we shall use frequently as a base of $\mathcal{N}(\mathbb{T})$ the sequence $(\mathbb{T}_n)$ where $\mathbb{T}_n = \{e^{2\pi i t} : \ t \in \mathbb{R}, |t| \leq 1/4n\}$ for every $n \in \mathbb{N}$. Clearly $\mathbb{T}_+ = \mathbb{T}_1$.

For an abelian topological group G, the group of all continuous homomorphisms $\chi : G \to \mathbb{T}$, usually called *continuous characters*, with pointwise multiplication, is denoted by $G^\wedge$ and called *character group* or *dual group* of G. We will often denote the constant character simply by 1, regardless of the group G.

Let G be an abelian topological group G; for $A \subset G$, the set $A^\triangleright := \{\chi \in G^\wedge : \ \chi(A) \subset \mathbb{T}_+\}$ is named the *polar* of A. In what follows, when polars are computed in a group without any topology it is implicitly assumed that the group is endowed with the discrete topology. In some important cases, this distinction is not necessary, see Lemma 2 below.

If $B \subset G^\wedge$, $B^\triangleleft := \{x \in G : \ \chi(x) \subset \mathbb{T}_+ \quad \forall \chi \in B\}$ is named the *inverse polar* of B. For $A \subset G$, we abbreviate $(A^\triangleright)^\triangleleft$ to $A^{\triangleright\triangleleft}$, and $(A^\triangleright)^\triangleright$ to $A^{\triangleright\triangleright}$. We will say that A is *quasi-convex* if $A = \bigcap_{\chi \in A^\triangleright} \chi^{-1}(\mathbb{T}_+)$, that is, $A = A^{\triangleright\triangleleft}$. Intersections of families of quasi-convex sets, as well as inverse images of quasi-convex sets under continuous homomorphisms, are quasi-convex sets. For an arbitrary set A, $A^{\triangleright\triangleleft}$ is the intersection of all quasi-convex sets containing A and thus we also call it the quasi-convex hull of A and we will often denote it by $\mathrm{qc}(A)$. A quasi-convex subset of an abelian group is simply a quasi-convex subset of the same group endowed with the discrete topology.

A topological abelian group is said to be *locally quasi-convex* if it has a basis of neighborhoods of the neutral element formed by quasi-convex sets (see e. g. [4, p. 2]).

A subset A of a topological abelian group G is said to be *precompact* or *totally bounded* in G if for every $U \in \mathcal{N}(G)$ there exists a finite subset $F \subset G$ with $A \subset F + U$ (where $F + U$ stands for $\{f + u : f \in F, u \in U\}$).

For any nonempty family $\mathfrak{S}$ of nonempty subsets of a topological group G, we may consider on the dual group $G^\wedge$ the topology $\mathfrak{T}_\mathfrak{S}$ of uniform convergence on members of $\mathfrak{S}$. If $\mathfrak{S}$ is well directed in the sense of [11], the family $\{A^\triangleright : A \in \mathfrak{S}\}$ forms a basis of neighborhoods of the neutral element for $(G^\wedge, \mathfrak{T}_\mathfrak{S})$. It is easy to see that the families of finite sets, compact sets and precompact sets, are well directed families and they induce on $G^\wedge$ the topologies of uniform convergence on finite sets ($\mathfrak{T}_p$), compact sets ($\mathfrak{T}_{co}$) and precompact sets ($\mathfrak{T}_{pc}$), respectively. We will use the following important fact:

Proposition 1. *([1, Prop. 3.5]) Let U be a neighborhood of the neutral element in the topological abelian group G. Then $U^\triangleright$ is a $\mathfrak{T}_{pc}$-compact subset of $G^\wedge$.*

For a subset U of an abelian group G, such that $0 \in U$, and a natural number n, we set $U_{(n)} := \{x \in G : x \in U,\ 2x \in U,\ \ldots,\ nx \in U\}$, and $U_{(\infty)} := \bigcap_{n\in\mathbb{N}} U_{(n)}$. It is known that $\mathbb{T}_n = (\mathbb{T}_+)_{(n)}$ for every $n \in \mathbb{N}$ (and hence $(\mathbb{T}_+)_{(\infty)} = \{1\}$); from this fact it is easy to derive the following useful property:

Lemma 2. *([24], Lemma 2.1) Let G be an abelian group and V a subset of G which contains the neutral element. Any homomorphism $\chi : G \to \mathbb{T}$ which sends V into $\mathbb{T}_+$ is continuous with respect to any group topology on G for which V is a neighborhood of the neutral element. Hence the polar of V*

is the same whenever computed with respect to any of these group topologies (e. g. the discrete topology).

3 A Schauder theorem for groups

In the following statement we recall the construction of the locally quasi-convex topologies associated with the quasi-convex subsets of a group. The similar construction is known for locally convex topologies and absolutely convex subsets of a vector space.

Lemma 3. *(cf. [2, Sect. 2]; see also [3, Cor. 2.6]) Let G be an abelian group and let U be a quasi-convex subset of G, then*

(a) the sequence $\{U_{(n)} : n \in \mathbb{N}\}$ is a basis of neighborhoods of the neutral element in G for a group topology $\mathcal{T}_U$ on G.

(b) $(G, \mathcal{T}_U)$ is a (not necessarily Hausdorff) locally quasi-convex topological abelian group. Moreover,

$$(G, \mathcal{T}_U)^{\wedge} = \bigcup_{n\in\mathbb{N}} (U_{(n)})^{\triangleright}.$$

(c) $U_{(\infty)}$ is a closed subgroup of $(G, \mathcal{T}_U)$; moreover, $U_{(\infty)}$ coincides with the closure of $\{0\}$ in $(G, \mathcal{T}_U)$.

(d) Put $G_U := (G/U_{(\infty)},\ \mathcal{T}_U/U_{(\infty)})$ and let φ_U be the canonical map from G to G_U. For every $n \in \mathbb{N}$ the set $\varphi_U(U_{(n)})$ coincides with $(\varphi_U(U))_{(n)}$, and this sequence is a basis of quasi-convex neighborhoods of the neutral element for the topology $\mathcal{T}_U/U_{(\infty)}$, which in particular coincides with $\mathcal{T}_{\varphi_U(U)}$.

(e) G_U is a metrizable locally quasi-convex topological abelian group, which is uniformly free from small subgroups in the sense of [14].

Remark 1. *Lemma 3(a) may not be true for an arbitrary symmetric subset U containing the neutral element, as the following example (pointed out to us by Prof. Elena Martín Peinador) shows. Let $G = \mathbb{R}^2$ and $U = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$. Then U is a symmetric set containing $(0, 0)$, but for no $m \in \mathbb{N}$ there exists $m' \in \mathbb{N}$ satisfying the relation $U_{(m')} + U_{(m')} \subset U_{(m)}$.*

Let G be a topological abelian group and U a nonempty subset of G. The subset $U^{\triangleright}$ is quasi-convex in $G^{\wedge}$, and the topology $\mathcal{T}_{U^{\triangleright}}$ on $G^{\wedge}$ coincides with the topology $\mathfrak{T}_{\{U\}}$ of uniform convergence on U. The Hausdorff group associated with $(G^{\wedge}, \mathcal{T}_{U^{\triangleright}})$ is the quotient $(G^{\wedge}/(U^{\triangleright})_{(\infty)}, \mathcal{T}_{U^{\triangleright}}/(U^{\triangleright})_{(\infty)})$. Observe that $(U^{\triangleright})_{(\infty)} = \{\chi \in G^{\wedge} : \chi(U) = \{1\}\}$. It follows that $\mathcal{T}_{U^{\triangleright}}$ is a Hausdorff topology provided that $\mathrm{gp}(U) = G$. (For a subset U of an abelian group G, we denote by $\mathrm{gp}(U)$ the subgroup of G generated by U.)

Lemma 4. *Let U be a quasi-convex subset of an abelian group G. The group homomorphism*

$$\rho_U : \mathrm{gp}(U) \to ((G, \mathcal{T}_U)^{\wedge}, \mathcal{T}_{U^{\triangleright}})^{\wedge}, \quad \rho_U(x)(\chi) = \chi(x)$$

$$\forall x \in \mathrm{gp}(U) \quad \forall \chi \in (G, \mathcal{T}_U)^{\wedge}$$

is well defined and induces an embedding

$$(\mathrm{gp}(U))_U \to ((G, \mathcal{T}_U)^{\wedge}, \mathcal{T}_{U^{\triangleright}})^{\wedge}_{U^{\triangleright\triangleright}}$$

Proof. Let us show that ρ_U is well defined, i. e., that for every $x \in \mathrm{gp}(U)$ the evaluation $\rho_U(x)$ on x is $\mathcal{T}_{U^{\triangleright}}$-continuous. Consider the evaluation map from G to $\mathrm{Hom}((G, \mathcal{T}_U)^{\wedge}, \mathbb{T})$. The inverse image of the subgroup $((G, \mathcal{T}_U)^{\wedge}, \mathcal{T}_{U^{\triangleright}})^{\wedge}$ by this map clearly contains U, hence it contains $\mathrm{gp}(U)$.

For $x \in \mathrm{gp}(U)$ and $n \in \mathbb{N}$,

$$\begin{aligned}\rho_U(x) \in (U^{\triangleright\triangleright})_{(n)} \quad &\Leftrightarrow \quad \forall \chi \in U^{\triangleright} \;\; \chi(x),\, \chi(2x), \ldots, \chi(nx) \in \mathbb{T}_+ \\ &\Leftrightarrow \quad x,\, 2x, \ldots, nx \in U^{\triangleright\triangleleft} \Leftrightarrow x,\, 2x, \ldots, nx \in U \Leftrightarrow x \in U_{(n)}.\end{aligned}$$

From this we deduce, for every $n \in \mathbb{N}$,

$$\rho_U(U_{(n)}) = \rho_U(\mathrm{gp}(U)) \cap (U^{\triangleright\triangleright})_{(n)}$$

and hence, ρ_U is continuous and open onto its image for the topologies $\mathcal{T}_U$ and $\mathcal{T}_{U^{\triangleright\triangleright}}$. Thus the corresponding map between the associated Hausdorff groups is continuous and open onto its image, too. Moreover it is injective, since the above equivalences $\rho_U(x) \in (U^{\triangleright\triangleright})_{(n)} \Leftrightarrow x \in U_{(n)}$ $(n \in \mathbb{N})$ imply $U_{(\infty)} = \rho^{-1}((U^{\triangleright\triangleright})_{(\infty)})$. $\square$

Recall that for any two topological abelian groups G and H and any continuous group homomorphism $u : G \to H$ the dual homomorphism $u^\wedge : H^\wedge \to G^\wedge$ is defined by $u^\wedge(\chi) = \chi \circ u$, for every $\chi \in H^\wedge$.

Lemma 5. *Suppose that G and H are abelian groups and V, U are quasi-convex subsets of G and H, respectively. Let $u : G \to H$ be a group homomorphism and suppose that $u(V) \subset U$. Then*

(a) $u : (G, \mathcal{T}_V) \to (H, \mathcal{T}_U)$ *is continuous.*

(b) The dual homomorphism $u^\wedge : ((H, \mathcal{T}_U)^\wedge, \mathcal{T}_{U^\triangleright}) \to ((G, \mathcal{T}_V)^\wedge, \mathcal{T}_{V^\triangleright})$ *satisfies* $u^\wedge(U^\triangleright) \subset V^\triangleright$ *and is therefore continuous.*

Proof. (a) $u(V) \subset U$ clearly gives $u(V_{(n)}) \subset U_{(n)}$ for every $n \in \mathbb{N}$, which implies that u is continuous with respect to the topologies $\mathcal{T}_V$ and $\mathcal{T}_U$.

(b) It is easily verified that $u^\wedge(U^\triangleright) \subset V^\triangleright$ holds, and the continuity of $u^\wedge$ follows as in (a).

$\square$

Definition 1. *Let G and H be topological abelian groups. A homomorphism $u : G \to H$ is said to be precompact (compact) if for some $V \in \mathcal{N}(G)$, the set $u(V)$ is precompact (relatively compact) in H.*

Next we present a version for topological groups of the well known theorem of Schauder for normed spaces as it appears for example in [21].

Theorem 6. *Let G and H be abelian groups, V and U quasi-convex subsets of G and H, respectively. Let $u : (G, \mathcal{T}_V) \to (H, \mathcal{T}_U)$ be a continuous group homomorphism such that $u(V) \subset U$ and let $u^\wedge : (H, \mathcal{T}_U)^\wedge \to (G, \mathcal{T}_V)^\wedge$ be the dual homomorphism.*

(a) *If u is a precompact homomorphism from $(G, \mathcal{T}_V)$ to $(H, \mathcal{T}_U)$ such that $u(V)$ is $\mathcal{T}_U$-precompact then $u^\wedge(U^\triangleright)$ is $\mathcal{T}_{V^\triangleright}$-compact and hence $u^\wedge$ is a compact homomorphism from $((H, \mathcal{T}_U)^\wedge, \mathcal{T}_{U^\triangleright})$ to $((G, \mathcal{T}_V)^\wedge, \mathcal{T}_{V^\triangleright})$.*

(b) *If $u^\wedge$ is a precompact homomorphism from $((H, \mathcal{T}_U)^\wedge, \mathcal{T}_{U^\triangleright})$ to $((G, \mathcal{T}_V)^\wedge, \mathcal{T}_{V^\triangleright})$ such that $u^\wedge(U^\triangleright)$ is $\mathcal{T}_{V^\triangleright}$-precompact, then $u(V)$ is $\mathcal{T}_U$-precompact and hence u is a precompact homomorphism from $(G, \mathcal{T}_V)$ to $(H, \mathcal{T}_U)$.*

Proof. (a) Consider on $(H, \mathcal{T}_U)^\wedge$ the topology $\mathfrak{T}_{pc}$. The assertion is a consequence of the following two facts:

(1) $U^\triangleright$ is a compact subset of $((H, \mathcal{T}_U)^\wedge, \mathfrak{T}_{pc})$. This follows at once from Prop. 1.

(2) $u^\wedge$ is continuous from $((H, \mathcal{T}_U)^\wedge, \mathfrak{T}_{pc})$ to $((G, \mathcal{T}_V)^\wedge, \mathcal{T}_{V^\triangleright})$. This is a consequence of the inclusions $u^\wedge((u(V)^\triangleright)_{(n)}) \subset (V^\triangleright)_{(n)}$ $(n \in \mathbb{N})$, since by hypothesis $u(V)$ is $\mathcal{T}_U$-precompact.

(b) By Lemma 5 we may define the (continuous) bidual homomorphism

$$u^{\wedge\wedge} : (((G, \mathcal{T}_V)^\wedge, \mathcal{T}_{V^\triangleright})^\wedge, \mathcal{T}_{V^{\triangleright\triangleright}}) \to (((H, \mathcal{T}_U)^\wedge, \mathcal{T}_{U^\triangleright})^\wedge, \mathcal{T}_{U^{\triangleright\triangleright}})$$

Since $u^\wedge(U^\triangleright)$ is $\mathcal{T}_{V^\triangleright}$-precompact and contained in $V^\triangleright$, (a) implies that $u^{\wedge\wedge}(V^{\triangleright\triangleright})$ is $\mathcal{T}_{U^{\triangleright\triangleright}}$-compact. Consider the maps ρ_V and ρ_U as in Lemma 4. Since $\rho_V(V) \subset V^{\triangleright\triangleright}$, it follows that $u^{\wedge\wedge}(\rho_V(V))$ is $\mathcal{T}_{U^{\triangleright\triangleright}}$-relatively compact. On the other hand it is clear that $u^{\wedge\wedge}(\rho_V(V)) = \rho_U(u(V))$. Since ρ_U induces an embedding of $(\mathrm{gp}(U))_U$ into $((H, \mathcal{T}_U)^\wedge, \mathcal{T}_{U^\triangleright})^\wedge_{U^{\triangleright\triangleright}}$ (Lemma 4), $u(V)$ is $\mathcal{T}_U$-precompact.

□

4 A characterization of Schwartz groups

Definition 2. *([2]) Let G be a Hausdorff topological abelian group. We say that G is a Schwartz group if for every neighborhood of the neutral element*

U in G there exists another neighborhood of the neutral element V in G and a sequence (F_n) of finite subsets of G such that

$$V \subset F_n + U_{(n)} \quad \textit{for every} \;\; n \in \mathbb{N}.$$

For later use we include the following statement which can be viewed as a group version of the known fact that a normed Schwartz space is finite-dimensional.

Proposition 7. *Let U be a quasi-convex subset of an abelian group G such that $(G, \mathcal{T}_U)$ is a Schwartz group, then $(G, \mathcal{T}_U)$ is a locally precompact topological abelian group.*

Proof. Since $(G, \mathcal{T}_U)$ is a Schwartz group and the sequence $(U_{(k)})_{k\in\mathbb{N}}$ is a basis for $\mathcal{N}(G, \mathcal{T}_U)$, for $U \in \mathcal{N}(G, \mathcal{T}_U)$ there exists a natural number k and a sequence (F_n) of finite subsets of G such that

$$U_{(k)} \subset F_n + U_{(n)} \quad \text{for every} \;\; n \in \mathbb{N}.$$

From this, and again using the fact that $(U_{(n)})_{n\in\mathbb{N}}$ is a basis for $\mathcal{N}(G, \mathcal{T}_U)$, we deduce that $U_{(k)}$ is precompact in G. □

Suppose that U and V are quasi-convex subsets of an abelian group G, such that $V_{(n)} \subset U$ for some $n \in \mathbb{N}$. Then we can consider the linking homomorphism

$$\varphi_{VU} : G_V \to G_U, \;\; \varphi_V(x) \mapsto \varphi_U(x)$$

which is clearly continuous.

Theorem 8. *Let G be a locally quasi-convex topological abelian group. The following are equivalent:*

(i) G is a Schwartz group.

(ii) For every quasi-convex $U \in \mathcal{N}(G)$ there exists $V \in \mathcal{N}(G)$ such that V is precompact in $(G, \mathcal{T}_U)$ (equivalently, $\varphi_U(V)$ is precompact in G_U).

(iii) For every quasi-convex $U \in \mathcal{N}(G)$ there exists a quasi-convex $V \in \mathcal{N}(G)$ such that $V \subset U$ and $\varphi_{VU} : G_V \to G_U$ is precompact.

(iv) For every quasi-convex $U \in \mathcal{N}(G)$ there exist a quasi-convex $V \in \mathcal{N}(G)$ such that $V \subset U$ and $U^{\triangleright}$ is precompact in $((G, \mathcal{T}_V)^\wedge, \mathcal{T}_{V^{\triangleright}})$ (equivalently, $\varphi_{V^{\triangleright}}(U^{\triangleright})$ is precompact in $((G, \mathcal{T}_V)^\wedge)_{V^{\triangleright}}$).

Proof. (i)⇔(ii): This is a direct consequence of the definition of a Schwartz group.

(ii)⇒(iii): Fix a quasi-convex $U \in \mathcal{N}(G)$. Fix $V \in \mathcal{N}(G)$ such that $\varphi_U(V)$ is precompact in G_U. We can suppose that V is quasi-convex and $V \subset U$. Since $\varphi_{VU}(\varphi_V(V)) = \varphi_U(V)$, we deduce that φ_{VU} is precompact.

(iii)⇒(ii): Fix a quasi-convex $U \in \mathcal{N}(G)$. Find a quasi-convex $V \in \mathcal{N}(G)$ satisfying the property described in (iii). For some $n \in \mathbb{N}, \varphi_{VU}(\varphi_V(V_{(n)})) = \varphi_U(V_{(n)})$ is precompact in G_U.

(ii)⇔ (iv): For quasi-convex $U,\ V \in \mathcal{N}(G)$ such that $V \subset U$, consider the identity map $(G, \mathcal{T}_V) \to (G, \mathcal{T}_U)$. This map and the sets V and U satisfy the hypothesis of Theorem 6; an application of this result gives the equivalence between (b) and (d). □

5 Convergence of characters on the dual of a Schwartz group

5.1 Continuous convergence

Let G be a topological abelian group and $G^\wedge$ its dual group. Let $\omega : G^\wedge \times G \to \mathbb{T}$ be the natural evaluation map. We say that a filter $\mathcal{F}$ in $G^\wedge$ converges *continuously* ([7, 0.2]) to an element $\chi \in G^\wedge$ if for every $x \in G$ and every filter $\mathcal{H}$ in G that converges to x, $\omega(\mathcal{F} \times \mathcal{H})$ converges to $\chi(x)$ in $\mathbb{T}$ (here, $\mathcal{F} \times \mathcal{H}$ denotes the filter generated by the products $F \times H$, where $F \in \mathcal{F}$, $H \in \mathcal{H}$, and $\omega(\mathcal{F} \times \mathcal{H})$ is the filter generated by the sets $\omega(F \times H) = \{f(x) : f \in F,\ x \in H\}$). It can be proved ([6, Prop. 8.1.8]) that $\mathcal{F}$ converges continuously to an element χ if for all $x \in G$, the filter $\mathcal{F}(x)$ converges to $\chi(x)$ and there exists some neighborhood U of the neutral element in G such that $\mathcal{F}$ contains $U^{\triangleright}$.

Continuous convergence can be defined also for nets. A net $(\chi_\alpha)_{\alpha \in A}$ in $G^\wedge$ converges continuously to an element $\chi \in G^\wedge$ if for every $x \in G$ and every net $(x_\beta)_{\beta \in B}$ in G which converges to x, the net $(\chi_\alpha(x_\beta))_{(\alpha,\beta) \in A \times B}$ converges to $\chi(x)$ in $\mathbb{T}$. A net in $G^\wedge$ converges continuously to a character χ if and only if its tail filter converges continuously to the same limit, if and only if $\chi_\alpha(x)$ converges to $\chi(x)$ for every $x \in G$ and there exist a neighborhood U of the neutral element in G and an index $\alpha_0 \in A$ such that $\chi_\alpha \in U^{\triangleright}$ for every $\alpha \geq \alpha_0$ (this is Lemma 2 in [10]).

Continuous convergence is compatible with the group structure of $G^\wedge$. In particular a filter $\mathcal{F}$ converges to $\chi \in G^\wedge$ continuously if and only if the filter $\mathcal{F}\chi^{-1}$ converges to 1 continuously. More information about this type of convergence can be found in [6].

The definition of continuous convergence means that it is the weakest convergence on $G^\wedge$ making the evaluation ω continuous. It is a well known fact that the continuous convergence is stronger than the uniform convergence on compact sets and if G is a locally compact group, they coincide.

5.2 Local uniform convergence

We say that a filter $\mathcal{F}$ in $G^\wedge$ converges *locally uniformly* to an element $\chi \in G^\wedge$ ([7, 5.4]) if it converges uniformly on some neighborhood of the neutral element, i. e. there exists a neighborhood V of the neutral element in G such that for all $n \in \mathbb{N}$, there exists some $S_n \in \mathcal{F}$ with $S_n(x) \subset \chi(x)\mathbb{T}_n$ for every $x \in V$.

Local uniform convergence can be defined also for nets. We say that $(\chi_\alpha)_{\alpha \in A}$ converges locally uniformly to an element $\chi \in G^\wedge$ if it converges uniformly on some neighborhood of the neutral element, i. e. there exists a neighborhood V of the neutral element in G such that for all $n \in \mathbb{N}$, there exists some $\alpha_n \in A$ with $\chi_\alpha(x) \subset \chi(x)\mathbb{T}_n$ for every $\alpha \geq \alpha_n$ and every

$x \in V$. This happens if and only if the tail filter of $(\chi_\alpha)_{\alpha\in A}$ converges locally uniformly according to the definition given above.

Local uniform convergence is compatible with the group structure of $G^\wedge$. Observe that if a filter $\mathcal{F}$ on $G^\wedge$ converges locally uniformly, then there exists a neighborhood U of the neutral element in G such that $\mathcal{F}$ converges in $\mathcal{T}_{U^\triangleright}$.

Definition 3. *Let G be a topological abelian group. We say that G is locally generated if* $\mathrm{gp}(U) = G$ *for every* $U \in \mathcal{N}(G)$.

It is known that a connected topological abelian group is locally generated; the converse is true in the locally compact case (see [20]).

Lemma 9. *Let G be a locally generated topological abelian group. Every locally uniformly convergent filter in G converges pointwise to the same limit.*

Proof. This is a consequence of the fact that if a filter of characters $\mathcal{F}$ converges uniformly to 1 on $U \in \mathcal{N}(G)$, in particular it converges pointwise to 1 on $\mathrm{gp}(U)$. □

Lemma 10. *Let G be a topological abelian group. If a filter $\mathcal{F}$ on $G^\wedge$ converges both pointwise and locally uniformly, then it converges continuously.*

Proof. Suppose that $\mathcal{F}$ converges to 1 locally uniformly and pointwise. There exists $U \in \mathcal{N}(G)$ such that $\mathcal{F}$ converges to 1 in $(G^\wedge, \mathcal{T}_{U^\triangleright})$, hence for every $m \in \mathbb{N}$ we have $(U^\triangleright)_{(m)} \in \mathcal{F}$; in particular $U^\triangleright \in \mathcal{F}$ and $\mathcal{F}$ converges continuously to 1. □

Note that, according to Lemma 9, the condition of pointwise convergence in Lemma 10 is redundant for locally generated groups.

In Subsection 5.4 we will see that within the class of locally quasi-convex groups, continuous convergence on the dual imply locally uniform convergence if and only if the original group is Schwartz.

5.3 A realization of the topology of locally generated Schwartz groups

Lemma 11. *Let G be a metrizable topological abelian group. For any precompact set $A \subset G$ there exists a sequence $\{x_n : n \in \mathbb{N}\}$ in G, which converges to the neutral element, such that $\{x_n : n \in \mathbb{N}\}^{\triangleright} \subset A^{\triangleright}$.*

Proof. Let $\widetilde{G}$ be the completion of G. The set $K = \overline{A}$ is compact in $\widetilde{G}$. By [1, 4.4], there exists a sequence $\{x_n : n \in \mathbb{N}\}$ in G, which converges to the neutral element, such that $\{x_n\}^{\triangleright} \subset K^{\triangleright}$. In particular $\{x_n\}^{\triangleright} \subset A^{\triangleright}$. (Note that the dual groups of G and $\widetilde{G}$ coincide.) $\square$

Proposition 12. *Let (G, τ) be a locally quasi-convex and locally generated Schwartz group. Then τ coincides with the topology of uniform convergence on the sequences of the dual group which converge locally uniformly to 1.*

Proof. Since G is locally quasi-convex, τ is the topology of uniform convergence on all equicontinuous subsets of the dual. The topology of uniform convergence on the sequences of the dual group which converge locally uniformly to 1, is weaker than τ. Inversely: fix a quasi-convex τ-neighborhood U; by Theorem 8 there exists V such that $\varphi_{V^{\triangleright}}(U^{\triangleright})$ is a precompact subset of the metrizable group $(G, \mathcal{T}_V)^{\wedge}_{V^{\triangleright}}$. Using Lemma 11, we find a sequence (χ_n) in $(G, \mathcal{T}_V)^{\wedge}$ which converges to the neutral element in $\mathcal{T}_{V^{\triangleright}}$, and its $\mathcal{T}_{V^{\triangleright}}$-polar is contained in the $\mathcal{T}_{V^{\triangleright}}$-polar of $U^{\triangleright}$. Let us show that $\{\chi_1, \chi_2, \cdots\}^{\triangleleft} \subset \mathrm{qc}(U)$ and, since U is quasi-convex, this will prove our statement. Note that, since V generates G, the evaluation map $\alpha(g)$ is $\mathcal{T}_{V^{\triangleright}}$-continuous for any $g \in G$. Suppose that $g \in \{\chi_1, \chi_2, \cdots\}^{\triangleleft}$. Then $\alpha(g)$ is in the $\mathcal{T}_{V^{\triangleright}}$-polar of $\{\chi_n\}$ and thus in the $\mathcal{T}_{V^{\triangleright}}$-polar of $U^{\triangleright}$. Hence $g \in \mathrm{qc}(U)$. $\square$

Proposition 12 is a group version of the implication (1)⇒(3) of [23, Th. 10.4.1].

5.4 Schwartz groups and convergence of characters

Theorem 13. *Let G be a locally quasi-convex group G. The following are equivalent:*

(i) G is a Schwartz group.

(ii) Every continuously convergent filter on $G^\wedge$ converges locally uniformly.

(iii) Every continuously convergent net on $G^\wedge$ converges locally uniformly.

Proof. (i)⇒(ii): We may restrict ourselves to filters which converge to 1. Let $\mathcal{F}$ be a filter on $G^\wedge$ which converges pointwise to 1, and contains $U^\triangleright$ for some $U \in \mathcal{N}(G)$. Since G is a Schwartz group, there exists another neighborhood of the neutral element V in G and a sequence of finite sets $S_k \subset G$ such that $V \subset S_k + U_{(k)}$ for every $k \in \mathbb{N}$. We will prove that $\mathcal{F}$ converges uniformly to 1 on V, that is, for every $n \in \mathbb{N}$ there exists $F \in \mathcal{F}$ with $F(V) \subset \mathbb{T}_n$. By pointwise convergence $W_n := \{\chi \in G^\wedge : \chi(S_{2n}) \subset \mathbb{T}_{2n}\}$ belongs to $\mathcal{F}$. Put $F_n = W_n \cap U^\triangleright \in \mathcal{F}$. We have $F_n(V) \subset F_n(S_{2n} + U_{(2n)}) \subset F_n(S_{2n})F_n(U_{(2n)}) \subset \mathbb{T}_{2n}\mathbb{T}_{2n} = \mathbb{T}_n$.

(ii)⇒(i): Fix a quasi-convex neighborhood U of the neutral element in G. Let $\mathcal{F}$ be the filter generated by the sets $(U_{(2)} \cup S)^\triangleright = (U_{(2)})^\triangleright \cap S^\triangleright$, where S runs through all finite subsets of G. The filter $\mathcal{F}$ converges pointwise to 1, and contains $(U_{(2)})^\triangleright$, hence it converges continuously to 1. By hypothesis there exists a quasi-convex $V \in \mathcal{N}(G)$, $V \subset U$, such that $\mathcal{F}$ converges uniformly to 1 on V. Thus for every $n \in \mathbb{N}$ there exist a finite set $S_n \subset G$ with $((U_{(2)})^\triangleright \cap S_n^\triangleright)(V) \subset \mathbb{T}_n$. Now consider the natural homomorphism $\phi : (G, \mathcal{T}_U)^\wedge \to (G, \mathcal{T}_V)^\wedge$. Let us show that the map

$$\phi|_{U^\triangleright} : (U^\triangleright, \mathfrak{T}_p|_{U^\triangleright}) \to ((G, \mathcal{T}_V)^\wedge, \mathcal{T}_{V^\triangleright})$$

is (uniformly) continuous. Given any $n \in \mathbb{N}$, for the finite set $S_n \subset G$ we have

$$\varphi_1,\ \varphi_2 \in U^\triangleright, \quad \varphi_1\varphi_2^{-1} \in S_n^\triangleright$$

$$\Rightarrow \quad \varphi_1\varphi_2^{-1} \in (U^\triangleright U^\triangleright \cap S_n^\triangleright) \subset (U_{(2)})^\triangleright \cap S_n^\triangleright$$
$$\Rightarrow \quad \varphi_1\varphi_2^{-1}(V) \subset \mathbb{T}_n \Rightarrow \varphi_1\varphi_2^{-1} \in (V^\triangleright)_{(n)}.$$

Since $U^\triangleright$ is $\mathfrak{T}_p$–compact (see [1, 3.5]), $U^\triangleright$ is $\mathcal{T}_{V^\triangleright}$–compact. Now Theorem 8 gives the desired result.

(ii)⇔(iii): This follows from the fact that a net of characters converges continuously (resp. locally uniformly) if and only if its tail filter converges continuously (resp. locally uniformly) to the same limit, and from the known construction ([25, Problem L to Chapter 2]) which associates to every abstract filter $\mathcal{F}$ a net whose tail filter is $\mathcal{F}$.

□

Corollary 14. *Let G a locally quasi-convex and locally generated group. Then G is a Schwartz group if and only if continuous convergence and locally uniform convergence coincide on $G^\wedge$.*

See [23, 10.4.1] for related results in topological vector space setting.

Let G be a topological abelian group. It is easy to see that a sequence $\{\chi_n\}$ in $G^\wedge$ converges continuously, as a net, according to the above definition, if and only if $\{\chi_n\}$ is equicontinuous and pointwise convergent.

In the next result we obtain a characterization of Schwartz groups in the class of metrizable groups, in terms of convergence of sequences of characters.

We denote by $c_0(\mathbb{T})$ the subgroup of $\mathbb{T}^\mathbb{N}$ formed by the sequences which converge to 1, endowed with the topology given by the complete invariant metric

$$d(t,s) = \sup_{n\in\mathbb{N}} |t_n - s_n|$$

Note that the set $U = \{t \in c_0(\mathbb{T}) : d(1,t) \leq \sqrt{2}\}$ is quasi-convex and $\mathcal{T}_U$ coincides with the topology of $c_0(\mathbb{T})$.

Theorem 15. *Let G be a metrizable separable locally quasi-convex group. The following statements are equivalent:*

(i) G is a Schwartz group.

(ii) Every continuously convergent sequence in $G^\wedge$ converges locally uniformly.

(iii) Every continuous homomorphism $\varphi : G \to c_0(\mathbb{T})$ is a compact homomorphism.

Proof. $(i) \Rightarrow (ii)$ follows from Theorem 13.

$(ii) \Rightarrow (i)$: We recall the following fact, which is true by general reasons: if G is a separable group, equicontinuous subsets of $G^\wedge$ are $\mathfrak{T}_p$-metrizable.

Fix $U \in \mathcal{N}(G)$. By Theorem 8, we need to find a quasi-convex $V \in \mathcal{N}(G)$ such that $V \subset U$ and $U^\triangleright$ is $\mathcal{T}_{V^\triangleright}$-precompact. Let $(V_k)_{k\in\mathbb{N}}$ be a decreasing basis of quasi-convex neighborhoods of the neutral element in G such that $V_1 \subset U$.

Suppose that $U^\triangleright$ is not precompact for any of the topologies $\mathcal{T}_{V_k^\triangleright}$. For every $k \in \mathbb{N}$ there exists a sequence $(\chi_i^{(k)})_{i\in\mathbb{N}}$ in $U^\triangleright$ without $\mathcal{T}_{V_k^\triangleright}$-convergent subsequences. Since $U^\triangleright$ is $\mathfrak{T}_p-$compact (see Prop. 1), we can suppose that each $(\chi_i^{(k)})$ converges pointwise to an element $\chi_k \in U^\triangleright$. Consider the sequences $\xi^{(k)} = (\xi_i^{(k)})_{i\in\mathbb{N}}$, defined by $\xi_i^{(k)} = \chi_i^{(k)}\overline{\chi_k}$, for every $k,\ i \in \mathbb{N}$. For every k, $\xi^{(k)}$ is contained in the equicontinuous (hence $\mathfrak{T}_p$-metrizable) set $W = U^\triangleright U^\triangleright$, converges to 1 in $\mathfrak{T}_p$, and does not contain any subsequence which converges in $\mathcal{T}_{V_k^\triangleright}$ to the constant character.

Let $(B_n)_{n\in\mathbb{N}}$ be a decreasing basis of neighborhoods of the neutral element for the topology $\mathfrak{T}_p$ in W. Define inductively a strictly increasing sequence of indices $(m_n)_{n\in\mathbb{N}}$ such that $\xi_i^{(k)} \in B_n$ for every $i \geq m_n$ and $1 \leq k \leq n$. We build up a new sequence joining successive blocks of the sequences $\xi^{(k)}$, in the following way:

$$\begin{aligned}
&(\xi^{(1)}_{m_1}, \xi^{(1)}_{m_1+1}, \ldots, \xi^{(1)}_{m_2-1},\\
&\xi^{(1)}_{m_2}, \xi^{(1)}_{m_2+1}, \ldots, \xi^{(1)}_{m_3-1}, \xi^{(2)}_{m_2}, \xi^{(2)}_{m_2+1}, \ldots, \xi^{(2)}_{m_3-1},\\
&\xi^{(1)}_{m_3}, \xi^{(1)}_{m_3+1}, \ldots, \xi^{(1)}_{m_4-1}, \xi^{(2)}_{m_3}, \xi^{(2)}_{m_3+1}, \ldots, \xi^{(2)}_{m_4-1}, \xi^{(3)}_{m_3}, \xi^{(3)}_{m_3+1}, \ldots, \xi^{(3)}_{m_4-1}, \ldots)
\end{aligned}$$

As $\xi^{(k)} \in W$ for every k, the above sequence is equicontinuous. Moreover, it converges to the neutral element in $\mathfrak{T}_p$, since the elements in the first block are in B_1, those in the second and the third ones are in B_2, those in the following three blocks are in B_3, and so on. Therefore, $(\xi^{(k)})$ converges to 1 continuously. On the other hand, $(\xi^{(k)})$ does not converge to the neutral element in any $\mathcal{T}_{V_k^\triangleright}$ since it contains tails of all the sequences $\xi^{(k)}$ as subsequences. This contradicts our hypothesis.

(ii) $\Leftrightarrow$ *(iii)*: For any sequence (χ_n) in $G^\wedge$ which converges pointwise to 1, the associated group homomorphism

$$x \in G \mapsto (\chi_n(x))_{n\in\mathbb{N}} \in c_0(\mathbb{T})$$

is continuous if and only if (χ_n) is equicontinuous (i. e. converges continuously to 1), and it is compact if and only if (χ_n) converges locally uniformly to 1. $\square$

Remark 2. *A non-Schwartz locally convex space E such that every continuous linear operator $T: E \to c_0$ is compact was found in [26, Example 3], thus answering a question posed in [12, p. 118]. The same example can be used in the group setting to show that implication $(iii) \Rightarrow (i)$ of Theorem 15 is not true in general for a non-metrizable G.*

Next we present a natural reformulation of $(i) \Leftrightarrow (ii)$ in Theorem 15 for sequentially barrelled groups. Recall that a sequentially barrelled group is a topological abelian group G such that every pointwise convergent sequence in $G^\wedge$ is equicontinuous ([28]). Every Baire group (in particular every complete and metrizable group) is sequentially barrelled.

Theorem 16. *Let G be a sequentially barrelled separable and metrizable locally quasi-convex group. The following statements are equivalent:*

(i) G is a Schwartz group.

(ii) Every pointwise convergent sequence in $G^\wedge$ converges locally uniformly.

Remark 3. *Theorem 16 was proved for Fréchet spaces by Floret ([15]). Our proof follows the presentation of this result given in [23, 11.6], where* *it was asked if the assumption of separability could be dropped. Later an affirmative answer was given independently by Bonet ([8]) and Lindström and Schlumprecht ([27]). We do not know whether the same is true in the group setting. Note that, unlike in the locally convex space case, there exist complete, metrizable, not separable locally quasi-convex Schwartz groups (e. g. any uncountable discrete group).*

Corollary 17. *Let G be an abelian group and $U \subset G$ a quasi-convex set. Assume that $(G, \mathcal{T}_U)$ is complete separable and Hausdorff. If every pointwise convergent sequence in $(G, \mathcal{T}_U)^\wedge$ converges uniformly on U, then $(G, \mathcal{T}_U)$ is locally compact.*

Proof. By (ii)⇒(i) in Theorem 16, $(G, \mathcal{T}_U)$ is a Schwartz group. By Proposition 7, $(G, \mathcal{T}_U)$ is locally precompact. Since it is complete, it is actually locally compact. □

Remark 4. *The above corollary is a group version of the famous Josefson-Nissenzweig theorem ([13, Ch. XII]). This theorem asserts the following: If for a Banach space any pointwise convergent sequence of continuous linear functionals converges in norm, then the space is finite-dimensional. We do not know whether Corollary 17 is true without the separability assumption.*

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