Do Investors Like to Diversify? A Study of Markowitz Preferences

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Abstract. We study rankings of completely and partially diversified portfolios and also of specialized assets when investors follow so-called Markowitz preferences. It turns out that diversification strategies for Markowitz investors are more complex than in the case of risk-averse and risk-inclined investors, whose investment strategies have been extensively investigated in the literature. In particular, we observe that for Markowitz investors, preferences toward risk vary depending on their sensitivities toward gains and losses. For example, it turns out that, unlike in the case of risk-averse and risk-inclined investors, Markowitz investors might prefer investing their entire wealth in just one asset. This finding helps us to better understand some financial anomalies and puzzles, such as the well known diversification puzzle, which notes that some investors tend to concentrate on investing in only a few assets instead of choosing the seemingly more attractive complete diversification.

Keywords and phrases: portfolio selection, diversified portfolio, Markowitz preferences, utility theory, risk aversion.
1 Introduction

The classical von Neuman and Morgenstern (1944) expected utility theory is based on the assumption that utility functions of risk-averse and risk-inclined investors are concave and convex, respectively; both are increasing. When examining the relative attractiveness of various forms of investments, Friedman and Savage (1948) observed that concave utility functions may not be best for explaining why investors buy lottery tickets or insurance. Markowitz (1952) addressed this concern by suggesting utility functions with convex segments on both the positive and the negative parts of the real line. Specifically, on the negative real half-line, the functions are initially convex and then concave, and then on the positive real half-line, they are convex and finally concave. Thus, the utility functions have three inflection points, with one of them being located at 0, which we call the status quo throughout the paper as it serves the borderline between losses and gains.

Subsequently, a number of authors departed from the original Markowitz’s suggested utility function and proposed functions that are convex for gains and concave for losses. Sometimes such functions are called reverse S-shaped, but throughout this paper we call them Markowitz utility functions, because this type of functions first appeared in Markowitz (1952). We call investors possessing Markowitz utility functions “Markowitz investors.” Such functions have only one inflection point, which is in concordance with a number of studies offering empirical support in favour of limiting the number of inflection points (e.g., Kraus and Litzenberger, 1976, Friend and Westerfield, 1980, Harvey and Siddique, 2000, Post and Levy, 2005, and reference therein).

Levy and Wiener (1998), Post and Levy (2005) study investors with Markowitz utility functions. Levy and Levy (2002, 2004), and Wong and Chan (2008) extend these works by developing a new criterion – called Markowitz stochastic dominance – which is designed for determining the dominance of one investment over another one for all Markowitz utility functions. In the present paper we continue this analysis. As we shall see in the following sections, the analysis of investor preferences is more complex for Markowitz utility functions than for (classical) concave or convex utility functions.

It is well known that when confronted with assets whose returns are independent and identical distributed (i.i.d.), risk-averse investors – that is, those whose utility functions
are concave – unanimously judge completely diversified portfolios as superior to partially diversified portfolios, which are in turn preferred by them to specialized assets. On the other hand, but under the same i.i.d. assumption, risk-inclined or, in other words, risk-seeking investors – that is, those whose utility functions are convex – prefer to invest in specialized assets than in partially diversified portfolios, which are in turn preferred by them to completely diversified portfolios.

There are very few articles addressing the portfolio diversification when investors possess Markowitz utility functions. In this paper we provide such an analysis. Specifically, in Section 2 we study rankings of diversified portfolios and specialized assets for Markowitz investors. In Section 3 we illustrate our main results and their optimality using examples and graphs. Section 4 concludes the paper.

2 Results

Assume that investors with certain initial endowments want to know shares of their endowments to be invested in $n$ risky assets whose returns denoted by $X_i$ are random. Hence, the investors want to maximize the expected utilities

$$E\left[u\left(\sum_{i=1}^{n} \alpha_i X_i\right)\right]$$

with respect to $\alpha_i \in [0, 1]$ such that $\sum_{i=1}^{n} \alpha_i = 1$. \footnote{We note that in this paper all investors are profit seeking, and thus all utility functions $u : \mathbb{R} \rightarrow \mathbb{R}$ are increasing.} This maximization problem is quite complex in general, and in this paper we therefore restrict ourselves to the analysis of whether it is better to completely diversify, partially diversify, or specialize portfolios; we call a portfolio

- **completely diversified** if $\alpha_i = 1/n$ for all $i = 1, \ldots, n$;
- **partially diversified** if $0 < \alpha_i < 1$ for at least one $i$;
- **specialized** if $\alpha_i = 1$ for only one $i$ and $\alpha_j = 0$ for all other $j \neq i$ (note that this type of portfolio consists of only one asset).
Diversification preferences for risk-averse and risk-inclined investors have been extensively studied (e.g., Li and Wong, 1999, and Wong, 2007) and the following proposition is well known.

**Proposition 2.1** Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables.

1) **Risk-inclined investors** prefer specialized portfolios to partially diversified ones, and prefer the latter ones to completely diversified portfolios. Namely, for every convex (and twice differentiable) utility function $u(x)$, we have that

$$
E\left[u\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\right] \leq E\left[u\left(\sum_{i=1}^{n} \alpha_i X_i\right)\right] \leq E\left[u\left(X_j\right)\right] \quad \text{for every } j \in \{1, \ldots, n\}. \quad (2.1)
$$

2) **Risk-averse investors** prefer completely diversified portfolios to partially diversified ones, and prefer the latter ones to specialized portfolios. Namely, for every concave (and twice differentiable) utility function $u(x)$, we have that

$$
E\left[u\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\right] \geq E\left[u\left(\sum_{i=1}^{n} \alpha_i X_i\right)\right] \geq E\left[u\left(X_j\right)\right] \quad \text{for every } j \in \{1, \ldots, n\}. \quad (2.2)
$$

We next study rankings of completely and partially diversified portfolios and also of specialized assets in the case of Markowitz investors, whose behavior turns out to be different from that of risk-averse and risk-inclined investors. Namely, their preferences toward risks vary depending on the sensitivity of their utility functions toward gains and losses. The following theorem states our findings rigorously.

**Theorem 2.1** Let $X_1, \ldots, X_n$ be i.i.d. and symmetric around zero random variables, and let the utility function $u(x)$ be twice differentiable. Then we have the following two statements:

1) If $u^{(2)}(-x) \geq -u^{(2)}(x)$ for all $x \in \mathbb{R}$, then inequalities (2.1) hold.

2) If $u^{(2)}(-x) \leq -u^{(2)}(x)$ for all $x \in \mathbb{R}$, then inequalities (2.2) hold.

The right-most bounds of both inequalities (2.1) and (2.2) hold without the assumption of identical distributions: we only need to require that the random variables $X_1, \ldots, X_n$ be independent, symmetric around zero, and have same means.
The proof of Theorem 2.1 is somewhat complex and we have therefore relegated it to an appendix. Note that verifying the bound $u^{(2)}(-x) \geq -u^{(2)}(x)$ for all $x \in \mathbb{R}$ is equivalent to verifying the same bound for only all $x \geq 0$. The same note holds for the bound $u^{(2)}(-x) \leq -u^{(2)}(x)$.

Theorem 2.1 implies that in the presence of i.i.d., symmetric, and mean zero assets, the diversification preferences of Markowitz investors can vary from preferring to diversify to preferring not to diversify depending on the sensitivity of their utility functions toward gains and losses. When, however, the sensitivities toward losses and gains are identical, then Theorem 2.1 implies that Markowitz investors can be indifferent toward any diversified portfolio and specialized asset. We formulate this observation as the following corollary.

**Corollary 2.1** Let $X_1, \ldots, X_n$ be i.i.d. and symmetric around zero random variables, and let the utility function $u(x)$ be twice differentiable. If the second derivative $u^{(2)}(x)$ is an odd function, that is, satisfies the equation $u^{(2)}(-x) = -u^{(2)}(x)$ for all $x \in \mathbb{R}$, then

$$E\left[u\left(\frac{\sum_{i=1}^n X_i}{n}\right)\right] = E\left[u\left(\sum_{i=1}^n \alpha_i X_i\right)\right] = E[u(X_j)] \quad \text{for every} \quad j \in \{1, \ldots, n\}. \quad (2.3)$$

The right-most equality of (2.3) holds assuming only that the random variables $X_1, \ldots, X_n$ are independent, symmetric around zero, and have same means.

Next we study Markowitz investors preferences when the status quo, which by definition separates losses from gains, is different from the mean-return of the assets. We can view this situation by setting the status quo to zero and vary the means $\mu$ of $X_i$’s. This is the approach that we have adopted in the following corollary.

**Corollary 2.2** Let $X_1, \ldots, X_n$ be i.i.d. and symmetric (around their means $\mu$) random variables. Let the utility function $u(x)$ be twice differentiable, and let its second derivative $u^{(2)}(x)$ be absolutely continuous with the third derivative (defined almost everywhere) $u^{(3)}(x)$ being non-negative for all $x \in \mathbb{R}$. Then we have the following two statements:

1) If $u^{(2)}(-x) \geq -u^{(2)}(x)$ for all $x \in \mathbb{R}$, then inequalities (2.1) hold whenever $\mu \geq 0$.

2) If $u^{(2)}(-x) \leq -u^{(2)}(x)$ for all $x \in \mathbb{R}$, then inequalities (2.2) hold whenever $\mu \leq 0$. 

The proof of Corollary 2.2 is somewhat involved and we have relegated it to an appendix. Note that the conditions on the second derivative $u^{(2)}(x)$ in parts 1) and 2) need to be verified only for all $x \geq 0$.

Corollary 2.2 implies that when facing i.i.d. and symmetric assets, Markowitz investors prefer not to diversify in the case of positive mean-returns of the underlying assets, but when the mean-returns are negative, then Markowitz investors prefer to diversify, assuming of course the formulated sensitivities of their utility functions to losses and gains.

3 An illustration

This section is designed to illustrate Theorem 2.1 and Corollary 2.2. In the case of Theorem 2.1 we shall use a skew-normal distribution (see Azzalini, 2005, and references therein) and vary its skewness parameter $\lambda$ in order to see when Theorem 2.1 fails and when it works. We shall also provide an analogous study of Corollary 2.2 but varying the mean $\mu$ of an underlying symmetric distribution. To make the two illustrations connected, in the case of Corollary 2.2 we shall use the same mean $\mu$ as that implied by the skew-normal distribution to be used for illustrating Theorem 2.1. We note at the outset that the skewness parameter $\lambda$ of the skew-normal distribution differs from the mean of the distribution, as we shall see in a formula below. Details follow.

With a sensitivity to gains and losses parameter $d > 0$, let the utility function be given by the formula

$$u(x) = \begin{cases} 
  x^3 & \text{if } x > 0, \\
  dx^3 & \text{if } x \leq 0,
\end{cases}$$

which we have visualized in Figure 3.1. The utility function is twice differentiable, its second derivative $u^{(2)}(x)$ is absolutely continuous, and we have the following expressions:

$$u^{(2)}(x) = \begin{cases} 
  6x & \text{if } x > 0, \\
  6dx & \text{if } x \leq 0,
\end{cases}$$

$$u^{(3)}(x) = \begin{cases} 
  6 & \text{if } x > 0, \\
  6d & \text{if } x < 0.
\end{cases}$$

We have chosen to work with the value $d = 0.1$ throughout this section because whenever $d \in (0, 1)$, then the bound $u^{(2)}(-x) \geq -u^{(2)}(x)$ holds for all $x \in \mathbb{R}$. Consequently, we are concerned with only the first parts of our results in the previous section, but analogous considerations are valid for the second parts as well.
Figure 3.1: The utility function $u(x)$ given by (3.1) with $d = 0.1$.

To facilitate calculating the three expected utilities, as well as visualizing them, we work with only two random variables, $X_1$ and $X_2$. Hence, $n = 2$ throughout this section. To explore the potential influence of skewness on Theorem 2.1, we conveniently assume that $X_1$ and $X_2$ are two independent random variables following the same skew-normal density (see Azzalini, 2005, and references therein)

$$f(x) = 2\phi(x)\Phi(\lambda x),$$

(3.3)

where $\phi$ and $\Phi$ are the standard normal density and distribution functions, respectively, and $\lambda \in \mathbb{R}$ is the skewness parameter. When $\lambda = 0$, then the density $f(x)$ reduces to the (symmetric) standard normal density $\phi(x)$. When $\lambda > 0$, then the density $f(x)$ is skewed to the right, and when $\lambda < 0$, then it is skewed to the left. The density function $f(x)$ is visualized in Figure 3.2 for the skewness parameter values $\lambda = -5, -1, 0, \text{ and } 5$, which we use in all our subsequent illustrations.\(^2\)

We consider the symmetric case first, that is, we set $\lambda = 0$, which is within the framework of Theorem 2.1. For any $\alpha \in [0, 1]$, the convex combination $\alpha X_1 + (1 - \alpha)X_2$ of the two random variables $X_1$ and $X_2$ is normally distributed with the mean 0 and the

\(^2\)We have included more negative values than positive ones because the first parts of our results in the previous section break down – as intended to illustrate – for random variables with negative skewness and/or negative means. For illustrating the second parts of the results, including more positive values would be warranted.
Figure 3.2: The skew-normal density function $f(x)$ for the the skewness parameter values $\lambda = -5$ (dotted with a negative mode), $-1$ (dashed), $0$ (solid), and $5$ (dotted with a positive mode).

variance $\alpha^2 + (1 - \alpha)^2$. Consequently,

$$
E[u(\alpha X_1 + (1 - \alpha)X_2)] = E[u(\sqrt{\alpha^2 + (1 - \alpha)^2} X_i)] \\
= (\alpha^2 + (1 - \alpha)^2)^{3/2}E[u(X_i)]
$$

(3.4)

for any $i = 1, 2$. Since the function $(\alpha^2 + (1 - \alpha)^2)^{3/2}$ defined on the interval $[0, 1]$ achieves its minimum at $\alpha = 1/2$ and identical maximums at the end-points of the interval $[0, 1]$, we have from equation (3.4) that

$$
E\left[u\left(\frac{1}{2} X_1 + \frac{1}{2} X_2\right)\right] \leq E[u(\alpha X_1 + (1 - \alpha)X_2)] \leq E[u(X_i)],
$$

(3.5)

as predicted by Theorem 2.1.

When $\lambda \neq 0$, then the random variables $X_1$ and $X_2$ are not symmetric, and thus Theorem 2.1 cannot be applied. To have a glimpse of what is happening in this case, in Figure 3.3 we have visualized the function

$$
H(\alpha) = E[u(\alpha X_1 + (1 - \alpha)X_2)], \quad 0 \leq \alpha \leq 1,
$$

in the case of the aforementioned four values of the skewness parameter $\lambda$. We see from Figure 3.3 that bounds of Theorem 2.1 can indeed fail when the symmetry of the underlying distribution is violated.

When the skewness changes, then the mean of the distribution also changes. This brings us into the framework of Corollary 2.2. To get a glimpse of how the mean of a sym-
metric distribution and the earlier discussed skewness might influence expected utilities, we continue working with the same three expected utilities (corresponding to completely diversified, partially diversified, and specialized portfolios) but now with two independent normal random variables $X_1$ and $X_2$, which are of course symmetric, with variances equal to 1 and their (identical) means given by the formula

$$\mu = \sqrt{\frac{2}{\pi}} \frac{\lambda}{\sqrt{1 + \lambda^2}}. \quad \text{(3.6)}$$

We continue using the aforementioned four values $\lambda = -5, -1, 0, \text{ and } 5$. Hence, $X_1$ and $X_2$ are two independent normal random variables $N(\mu, 1)$ with their mean values $\mu = -0.78239, -0.56419, 0, \text{ and } 0.78239$. (Note that $\mu$ in equation (3.6) is the mean of the skew-normal density (3.3).) In Figure 3.4 we have visualized the function $H(\alpha)$. We see from the figure that the inequalities of the first statement of Corollary 2.2 can indeed fail when the non-negativity assumption on the mean $\mu$ is violated.
4 Conclusions

We have developed a theoretical study that aids in explaining the attitudes of Markowitz investors toward risk, including their preferences for specialized, partially diversified, and completely diversified portfolios. Our main results show that unlike in the case of risk-averse and risk-inclined investors, Markowitz investors might prefer investing their entire wealth in just one asset. This finding helps us to better understand some financial anomalies and puzzles, including the well known diversification puzzle (e.g., Statman, 2004) which notes the tendency of some investors not to completely diversify investments and choose investing only in a few assets.

We have also provided an illustration showing that certain assumptions imposed for the validity of our results cannot in general be disposed of. Naturally, a further study exploring the validity boundaries of our results under additional distributional assumptions on the portfolio and/or asset returns would be of interest, including a study of portfolio rankings under constrains on the variability of portfolio and/or asset returns.
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References


\[\text{Appendix: Proofs}\]

Here we prove Theorem 2.1 and Corollary 2.2, for which we first establish a technical result formulated as Proposition A.1.

**Proposition A.1** Let \(X_1, \ldots, X_n\) be independent and symmetric around zero random variables, and let the utility function \(u(x)\) be twice differentiable. Then we have the following two statements:

1) If \(u^{(2)}(-x) \geq -u^{(2)}(x)\) for all \(x \in \mathbb{R}\), then

\[
E\left[u\left(\sum_{i=1}^{n} \alpha_i X_i\right)\right] \leq \max\{E[u(X_j)], j = 1, \ldots, n\}.
\]  
(A.1)

2) If \(u^{(2)}(-x) \leq -u^{(2)}(x)\) for all \(x \in \mathbb{R}\), then

\[
E\left[u\left(\sum_{i=1}^{n} \alpha_i X_i\right)\right] \geq \min\{E[u(X_j)], j = 1, \ldots, n\}.
\]  
(A.2)

**Proof of Proposition A.1.** We first prove the theorem in the case \(n = 2\) and then apply an induction argument to prove it for any \(n\). We have that

\[
\left(\frac{d}{d\alpha_1}\right)^2 E[u(\alpha_1 X_1 + (1 - \alpha_1) X_2)]
= \frac{1}{2} E\left[(X_1 - X_2)^2 \left(u^{(2)}(\alpha_1 X_1 + (1 - \alpha_1) X_2) + u^{(2)}(-\alpha_1 X_1 - (1 - \alpha_1) X_2)\right)\right],
\]  
(A.3)

where the equation holds due to the independence and symmetry of \(X_1\) and \(X_2\), which imply that the random variable \(\alpha_1 X_1 + (1 - \alpha_1) X_2\) is symmetric around 0. Note that
the right-hand side of equation (A.3) is non-negative because \( u^{(2)}(x) \geq -u^{(2)}(-x) \) for all \( x \in \mathbb{R} \) by assumption. Hence, the expectation \( E[u(\alpha_1 X_1 + (1 - \alpha_1)X_2)] \) is a convex function of \( \alpha_1 \in [0, 1] \) and its maximum is therefore achieved at either \( \alpha_1 = 0 \) or \( \alpha_1 = 1 \). This proves that

\[
E[u(\alpha_1 X_1 + (1 - \alpha_1)X_2)] \leq \max \{ E[u(X_1)], E[u(X_2)] \}. \tag{A.4}
\]

To proceed, we use an induction argument. Namely, for a given \( k \geq 2 \), let the bound

\[
E \left[u \left( \sum_{i=1}^{j} \alpha_i X_i \right) \right] \leq \max \{ E[u(X_i)], i = 1, \ldots, j \}. \tag{A.5}
\]

hold for every \( j \in \{2, \ldots, k\} \), all utility functions \( u(x) \) such that \( u^{(2)}(-x) \geq -u^{(2)}(x) \) for all \( x \in \mathbb{R} \) and all non-negative \( \alpha_i \)’s such that \( \sum_{i=1}^{j} \alpha_i = 1 \). We shall prove that bound (A.5) also holds when \( j = k + 1 \) for all utility functions \( u(x) \) such that \( u^{(2)}(-x) \geq -u^{(2)}(x) \) for all \( x \in \mathbb{R} \) and all non-negative \( \alpha_i \)’s such that \( \sum_{i=1}^{k+1} \alpha_i = 1 \). To this end, we write

\[
E \left[u \left( \sum_{i=1}^{k+1} \alpha_i X_i \right) \right] = E[u((1 - \alpha_{k+1})Y + \alpha_{k+1}X_{k+1})], \tag{A.6}
\]

where \( Y = \sum_{i=1}^{k} \beta_i X_i \) with \( \beta_i = \alpha_i/(1 - \alpha_{k+1}) \) for \( 1 \leq i \leq k \). Without loss of generality we can and thus do assume that \( \alpha_{k+1} < 1 \); otherwise all the other \( \alpha_i \)’s are equal to 0, which is a trivial case. Since \( X_i \)’s are independent and symmetric around zero, the random variable \( Y \) is symmetric around 0 and independent of \( X_{k+1} \). Thus by our inducational assumption (case \( j = 2 \)) we have that

\[
E[u((1 - \alpha_{k+1})Y + \alpha_{k+1}X_{k+1})] \leq \max \{ E[u(Y)], E[u(X_{k+1})] \}. \tag{A.7}
\]

Furthermore, by our inducational assumption (case \( j = k \)) implies that

\[
E[u(Y)] = E \left[u \left( \sum_{i=1}^{k} \beta_i X_i \right) \right] \leq \max \{ E[u(X_j)], j = 1, \ldots, k \}. \tag{A.8}
\]

Combining bounds (A.6), (A.7) and (A.8), we obtain that

\[
E \left[u \left( \sum_{i=1}^{k+1} \alpha_i X_i \right) \right] \leq \max \{ E[u(X_j)], j = 1, \ldots, k + 1 \}.
\]

This concludes the proof of Proposition A.1. \( \blacksquare \)
Proof of Theorem 2.1. The right-most inequalities of both parts 1) and 2) follow directly from Proposition A.1. We next establish the bound
\[
E \left[ u \left( \sum_{i=1}^{n} \frac{X_i}{n} \right) \right] \leq E \left[ u \left( \sum_{i=1}^{n} \alpha_i X_i \right) \right]
\] (A.9)
under the conditions of part 1) of Theorem 2.1. We start with the case \( n = 2 \). To see where the expectation \( E[u(\alpha_1 X_1 + (1-\alpha_1)X_2)] \) achieves its minimum with respect to \( \alpha_1 \), we equate its first derivative to zero:
\[
E \left[ (X_1 - X_2)u^{(1)}(\alpha_1 X_1 + (1-\alpha_1)X_2) \right] = 0.
\] (A.10)
One of the solutions to this equation is \( \alpha_1 = 1/2 \), and since the expectation is a convex function of \( \alpha_1 \) (recall statement (A.3) and the therein noted non-negativity of its right-hand side), the value \( \alpha_1 = 1/2 \) is a global minimum of \( E[u(\alpha_1 X_1 + \alpha_2 X_2)] \). Consequently, we have the bound
\[
E \left[ u \left( \frac{X_1}{2} + \frac{X_2}{2} \right) \right] \leq E \left[ u(\alpha_1 X_1 + \alpha_2 X_2) \right].
\] (A.11)
Assume now that, for a given \( k \geq 2 \), the bound
\[
E \left[ u \left( \sum_{i=1}^{j} \frac{X_i}{j} \right) \right] \leq E \left[ u \left( \sum_{i=1}^{j} \alpha_i X_i \right) \right]
\] (A.12)
holds for every \( j \in \{2, \ldots, k\} \), all utility functions \( u(x) \) such that \( u^{(2)}(-x) \geq -u^{(2)}(x) \) for all \( x \in \mathbb{R} \) and all non-negative \( \alpha_i \)'s such that \( \sum_{i=1}^{j} \alpha_i = 1 \). We shall prove that bound (A.12) also holds when \( j = k+1 \) for all utility functions \( u(x) \) such that \( u^{(2)}(-x) \geq -u^{(2)}(x) \) for all \( x \in \mathbb{R} \) and all non-negative \( \alpha_i \)'s such that \( \sum_{i=1}^{k+1} \alpha_i = 1 \). For this, we rewrite equation (A.6) as follows:
\[
E \left[ u \left( \sum_{i=1}^{k+1} \alpha_i X_i \right) \right] = E \left[ v \left( \sum_{i=1}^{k} \beta_i X_i \right) \right],
\] (A.13)
where \( \beta_i = \alpha_i/(1 - \alpha_{k+1}) \) for \( 1 \leq i \leq k \), and
\[
v(y) = E[u((1 - \alpha_{k+1})y + \alpha_{k+1}X_{k+1})].
\]
Without loss of generality, we can and thus do assume that \( \alpha_{k+1} < 1 \). In order to apply the above inductive assumption on the right-hand side of equation (A.6), we need to
check that \( v^{(2)}(x) + v^{(2)}(-x) \geq 0 \). This can be done as follows:

\[
v^{(2)}(x) + v^{(2)}(-x) = (1 - \alpha_{k+1})^2 \left( \mathbb{E}[u^{(2)}((1 - \alpha_{k+1})y + \alpha_{k+1}X_{k+1})] + \mathbb{E}[u^{(2)}(-(1 - \alpha_{k+1})y + \alpha_{k+1}X_{k+1})] \right)
= (1 - \alpha_{k+1})^2 \mathbb{E}[u^{(2)}((1 - \alpha_{k+1})y + \alpha_{k+1}X_{k+1}) + u^{(2)}(-(1 - \alpha_{k+1})y - \alpha_{k+1}X_{k+1})],
\]

where the second equation holds due to the symmetry of \( X_{k+1} \). Note that the right-hand side is non-negative because \( u^{(2)}(-x) \geq -u^{(2)}(x) \) for all \( x \in \mathbb{R} \) by assumption. Consequently, our inducational assumption can be applied, and we thus have that

\[
\mathbb{E}\left[ v\left( \sum_{i=1}^{k} \beta_i X_i \right) \right] \geq \mathbb{E}\left[ v\left( \sum_{i=1}^{k} \frac{X_i}{k} \right) \right] = \mathbb{E}\left[ u\left( (1 - \alpha_{k+1}) \sum_{i=1}^{k} \frac{X_i}{k} + \alpha_{k+1}X_{k+1} \right) \right].
\]

Denote the right-hand side of (A.14) by \( h(\alpha_{k+1}) \). Differentiating \( h(\alpha_{k+1}) \) with respect to \( \alpha_{k+1} \) and equating the derivative to 0, we obtain that

\[
\mathbb{E}\left[ \left( X_{k+1} - \sum_{i=1}^{k} \frac{X_i}{k} \right) u^{(1)} \left( (1 - \alpha_{k+1}) \sum_{i=1}^{k} \frac{X_i}{k} + \alpha_{k+1}X_{k+1} \right) \right] = 0.
\]

(A.15)

Since \( X_i \)'s are i.i.d. random variables, the choice \( \alpha_{k+1} = 1/(k+1) \) satisfies equation (A.15), which implies that \( \alpha_{k+1} = 1/(k+1) \) is a critical point of the function \( h(\alpha_{k+1}) \). We need to verify that the critical point is a global minimum of the function \( h(\alpha_{k+1}) \) with respect to \( \alpha_{k+1} \in [0,1] \). For this, we show that the function \( h(\alpha_{k+1}) \) is convex, which follows if its second derivative

\[
h^{(2)}(\alpha_{k+1}) = \mathbb{E}\left[ \left( X_{k+1} - \sum_{i=1}^{k} \frac{X_i}{k} \right)^2 u^{(2)} \left( (1 - \alpha_{k+1}) \sum_{i=1}^{k} \frac{X_i}{k} + \alpha_{k+1}X_{k+1} \right) \right]
\]

is non-negative. Since \( \sum_{i=1}^{k} X_i/k \) and \( X_{k+1} \) are independent and symmetric around 0, the non-negativity of \( h^{(2)}(\alpha_{k+1}) \) follows from a similar argument employed in the proof of Theorem 2.1 but now using the assumption \( u^{(2)}(-x) \geq -u^{(2)}(x) \) for all \( x \in \mathbb{R} \). Consequently, we have that \( \alpha_{k+1} = 1/(k+1) \) is a global minimum of the function \( h(\alpha_{k+1}) \) on the interval \([0,1]\). In view of this observation and since bound (A.14) is achieved with \( \beta_i = 1/k \) for all \( i = 1, \ldots, k \), which is equivalent to setting \( \alpha_i = 1/(k+1) \), we have bound (A.12) for \( j = k+1 \). This completes the proof of part 1). Part 2) of Theorem 2.1 follows analogously. This finishes the entire proof of Theorem 2.1. ■
**Proof of Corollary 2.2.** The corollary reduces to Theorem 2.1 with the utility function

\[ v(x) = u(x + \mu). \]

Hence, to prove part 1), we need to check that \( v^{(2)}(-x) \geq -v^{(2)}(x) \) for all \( x \in \mathbb{R} \). The condition can be written as follows:

\[ u^{(2)}(-x + \mu) \geq -u^{(2)}(x + \mu). \quad (A.17) \]

Since \( u^{(2)}(x + \mu) \geq -u^{(2)}(-x - \mu) \) due to the bound \( u^{(2)}(-z) \geq -u^{(2)}(z) \) for all \( z \in \mathbb{R} \), we have that

\[
\begin{align*}
&u^{(2)}(-x + \mu) + u^{(2)}(x + \mu) \geq u^{(2)}(-x + \mu) - u^{(2)}(-x - \mu) \\
&= \int_{-\mu}^{\mu} u^{(3)}(-x + y)dy. 
\end{align*}
\]

(A.18)

Since \( \mu \geq 0 \) and \( u^{(3)}(z) \geq 0 \) for all \( z \in \mathbb{R} \), the right-hand side of (A.18) is non-negative, thus proving bound (A.17) and establishing part 1) of Corollary 2.2.

To prove part 2), we need to check that \( v^{(2)}(-x) \leq -v^{(2)}(x) \) for all \( x \in \mathbb{R} \). The condition can be rewritten as follows:

\[ u^{(2)}(-x - \mu^*) \leq -u^{(2)}(x + \mu^*), \quad (A.19) \]

where \( \mu^* = -\mu \geq 0 \). Since \( u^{(2)}(-x - \mu^*) \leq -u^{(2)}(x + \mu^*) \) due to \( u^{(2)}(-z) \leq -u^{(2)}(z) \) for all \( z \in \mathbb{R} \) by assumption, we have that

\[
\begin{align*}
&u^{(2)}(-x - \mu^*) + u^{(2)}(x - \mu^*) \leq -u^{(2)}(x + \mu^*) + u^{(2)}(x - \mu^*) \\
&= -\int_{-\mu^*}^{\mu^*} u^{(3)}(x + y)dy. 
\end{align*}
\]

(A.20)

Since \( \mu^* \geq 0 \) and \( u^{(3)}(z) \geq 0 \) for all \( z \in \mathbb{R} \), the right-hand side of (A.20) is non-positive and so bound (A.19) holds. This establishes part 2) of Corollary 2.2 and completes the entire proof. \( \blacksquare \)