Structural and Multidisciplinary Optimization manuscript No.
(will be inserted by the editor)

Topology optimization of continuum structures with local and global stress constraints

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Received: date / Accepted: date

Abstract Topology structural optimization problems have been usually stated in terms of a maximum stiffness (minimum compliance) approach. The objective of this type of approach is to distribute a given amount of material in a certain domain, so that the stiffness of the resulting structure is maximized (that is, the compliance, or energy of deformation, is minimized) for a given load case. Thus, the material mass is restricted to a predefined percentage of the maximum possible mass, while no stress or displacement constraints are taken into account. This paper presents a different strategy to deal with topology optimization: a minimum weight with stress constraints Finite Element formulation for the topology optimization of continuum structures. We propose two different approaches in order to take into account stress constraints in the optimization formulation. The local approach of the stress constraints imposes stress constraints at predefined points of the domain (*i.e.* at the central point of each element). On the contrary, the global approach only imposes one global constraint that gathers the effect of all the local constraints by means of a certain so-called aggregation function. Finally, some application examples are solved with both formulations in order to compare the obtained solutions.

Keywords Topology optimization · Finite Element Method · minimum weight · stress constraints · local constraints · constraints aggregation

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1 Introduction

Ever since Schmit [29] proposed the revolutionary idea of designing minimum cost objects or systems by means of Mathematical Programming techniques, the optimization field has experienced a continuous evolution up to nowadays. This idea was first introduced in structural analysis by means of sizing and/or shape optimization formulations and, more recently, by means of topology optimization formulations [1].

The main idea of topology optimization of structures is to obtain the optimal distribution of material in a predefined domain.

In the original statement of this problem the unknown describing the material distribution is a binary function associated to solid/void configurations at each point of the domain when the domain is discretized (*i.e.* a FEM mesh is introduced). This approach leads to discrete programming type problems with a large number of design variables (*i.e.* one discrete design variable per element indicating if the cell is either solid or void). At present time there are no effective universal methods for solving this kind of problems, which makes this approach unaffordable in practice.

For this reason, the material distribution is normally described in terms of a continuum function that can take all the possible values between the two discrete limits: 0 (what indicates a void configuration at a given point) and 1 (what indicates a solid configuration at a given point). The value of this continuum function is referred to as the relative density in the vicinity of each point.

However, at this point it is necessary to define the structural response of the material being used for intermediate values of the relative density. This issue has been traditionally addressed by applying homogeniza-

tion techniques [1,2,3,13,33] over predefined microstructures of material (SIMP, Hole-in-cell, Rank-2-layered,...). Then, since the former discrete approach has been replaced by the new continuum approach, the resulting optimization problems can be tackled by means of more effective Mathematical Programming algorithms, as the once typically used in structural sizing and shape optimization.

Sizing and shape structural optimization problems have been mainly written in terms of minimum weight formulations with non-linear constraints. These constraints usually limit the maximum allowable stresses and displacements [14,19,32]. However, ever since Bendsøe and Kikuchi proposed the basic concepts of topology structural optimization in 1988 [1], most of these problems have been usually stated in terms of minimum compliance (maximum stiffness) approaches. Essentially, in this kind of formulations a given amount of material must be distributed within a given domain while the stiffness of the resulting structure is maximized (the compliance is minimized) for a given load case [1,2,3, 26].

The traditional minimum compliance formulations offer some obvious advantages, since one avoids dealing with a large number of highly non-linear constraints. This could be considered crucial, if one takes into account the large number of design variables that is inherent to this technique in real engineering applications.

However, minimum compliance formulations present several important drawbacks that could be considered as relevant as the advantages that they offer. Thus, multiple load cases can not be considered, and different solutions are obtained for different restrictions on the amount of material (sometimes with checkerboard layouts) and the final design could be unfeasible in practice since stress and displacement constraints are not imposed at all. Moreover, from a mathematical point of view, the minimum compliance approach is an "ill-posed problem", since the solution oscillates as the discretization refinement increases [3,17]. Different strategies have been proposed in the search of a proper procedure to overcome these difficulties. Most of them include the use of porous materials [2,3] (what calls again for the concept of relative density that was mentioned above)

The Solid Isotropic Material with Penalty (SIMP) formulation [2,3,18,27,28] is the most widely used minimum compliance approach at present time. In this formulation, a dimensionless design variable per element is introduced: the so-called "relative density" of an element. This is the complement to one of the porosity and its value ranges from 0 to 1. Thus, the aim is to obtain the design variables (and consequently the amount of porous material that must be distributed within each element) by minimizing a highly non-linear objective function (the compliance or deformation energy). The design is subjected to a single linear constraint, which is the total amount of material to be used. The total amount of material is defined by a fixed percentage of the total domain volume. This percentage is usually called the "filling factor".

On the other hand, the SIMP formulation is easy to implement in a FEM code for structural analysis. Moreover, several explicit procedures have been proposed to solve the optimization problem efficiently.

Nevertheless, the minimum compliance results may be questionable, since the final design depends on so many arbitrary parameters (filling factor, degree of discretization, applied penalization and stabilization techniques, image filtering processes, etc.)

On the other hand, since the most of structural design problems include stress and displacement limitations, it seems quite obvious that these criteria should be mandatorily considered in structural topology optimization formulations. In this paper, we follow this strategy and we propose a different approach that allows the consideration of stress and/or displacement constraints.

The most intuitive way of taking into account stress and/or displacement limitations consists of stating one (or several) constraint(s) of this type at selected points of the structure. The usual option is to consider stress constraints at a given point within each finite element of the mesh (what is usually referred to as the "local constraints" approach). This formulation has been also applied in the works of Duysinx [6,7], Pereira [25] or Yang [32], for example.

Although it presents some difficulties (which are fairly easy to solve), this formulation is very robust and the solutions obtained are very realistic. In addition, no artificial techniques are required to obtain adequate solutions. However, it also presents some unexpected numerical effects when the relative density tends to zero (*e.g.* the so-called singularity phenomena). Some of these effects can be easily explained from a theoretical point of view [5,11]. Therefore, it is necessary to introduce some modifications in the numerical algorithm in order to relax the singular theoretical solutions. These modifications are not unique and several relaxed approaches have been proposed in order to avoid the singularity phenomena [5,6]. However, these relaxation approaches produce highly non-linear functions and give rise to more complicated problems.

On the other hand, it is necessary to take into account that this approach may require very large computing resources. Moreover, the optimization problem

becomes, in fact, not only more time consuming but also more complicated from a numerical point of view when the number of non-linear constraints is very large.

A number of different techniques have been proposed in an attempt to reduce the number of stress constraints of the problem, due to the extensive computing effort that is required to obtain optimal solutions as the number of elements is increased. One of these approaches consists of stating one or several global functions, which include all the constraints of the local approach. Thus, the optimization problem becomes easier to solve since the number of constraints is drastically reduced. This technique is more recent than the local approach and it is usually referred to as the "global stress constraints approach".

Now the problem is to obtain a function that could correctly aggregate all the local constraints while presenting good numerical stability condition. In addition, it should be easily derived and it should adequately represent the whole set of local constraints. This is clearly the keypoint of this approach. Following these ideas, several contributions using global stress functions have been published (see for example [8]).

In this paper, we present a FEM based minimum weight with stress constraints (MWSC) approach for structural topology optimization problems. We have developed two different approaches: a local constraints formulation and a global constraint formulation based on the Kreisselmeier-Steinhauser function [16]. Finally, we present some application examples that compare the results obtained with the global and the local approaches.

2 The structural analysis model

2.1 The Structural Analysis Problem

Let Ω^o be a domain occupied by a deformed body. Due to the applied external loads, the initial body is deformed to a new one that corresponds to a new deformed domain Ω. Thus, every point P^o of the initial domain Ω^o is moved into a different position P in the deformed domain Ω. Let $\boldsymbol{r}^o$ and $\boldsymbol{r}$ be the coordinates of points P^o and P. Thus, the goal is to obtain the displacements

$$\boldsymbol{u}(\boldsymbol{r}^o) = \boldsymbol{r}(\boldsymbol{r}^o) - \boldsymbol{r}^o, \tag{1}$$

where $\boldsymbol{r}(\boldsymbol{r}^o)$ are the final coordinates of the point $\boldsymbol{P}^o$ once the domain is deformed.

Assuming the linear elasticity hypothesis, which implies small displacements and small displacement gradients, the associated strains $\boldsymbol{\varepsilon}(\boldsymbol{r}^o)$ and stresses $\boldsymbol{\sigma}(\boldsymbol{r}^o)$ can be obtained as

$$\boldsymbol{\varepsilon} = \boldsymbol{L}\boldsymbol{u}, \qquad \boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}, \tag{2}$$

where $\boldsymbol{L}$ is the differential operator that gives the strains for known values of the displacements ($\boldsymbol{u}$) and $\boldsymbol{D}$ is the constitutive matrix of the material [10].

On the other hand, let $d\Omega$ be the volume of a differential region in the vicinity of the point P^o. By definition, the volume occupied within the differential region is $d\Omega$. Therefore, the structural analysis problem can be written as

$$\begin{aligned}
&\textit{Find} && \boldsymbol{u} \in \mathcal{H}_u \\
&\textit{such that} && a(\boldsymbol{w},\boldsymbol{u}) = (\boldsymbol{w},\boldsymbol{b})_{\Omega^o} + (\boldsymbol{w},\boldsymbol{t})_{\Gamma^o_\sigma} \quad \forall \boldsymbol{w} \in \mathcal{H}_w \\
&\textit{being} && a(\boldsymbol{w},\boldsymbol{u}) = \iiint_{\Omega^o} (\boldsymbol{L}\boldsymbol{w})^T \boldsymbol{D}(\boldsymbol{L}\boldsymbol{u})\; d\Omega, \\
& && (\boldsymbol{w},\boldsymbol{b})_{\Omega^o} = \iiint_{\Omega^o} \boldsymbol{w}^T\boldsymbol{b}\; d\Omega, \\
& && (\boldsymbol{w},\boldsymbol{t})_{\Gamma^o_\sigma} = \iint_{\Gamma^o_\sigma} \boldsymbol{w}^T\boldsymbol{t}\; d\Gamma
\end{aligned} \tag{3}$$

where $\boldsymbol{u}$ and $\boldsymbol{w}$ are the trial and the test functions respectively. $\mathcal{H}_u$ is the subspace of feasible trial functions, $\mathcal{H}_w$ is the subspace of feasible test functions, $\boldsymbol{b}$ represents forces per unit of volume in the domain Ω^o and $\boldsymbol{t}$ represents forces per unit of area on the surface Γ^o.

2.2 The Structural Analysis Problem with Relative Density

Now, let the domain Ω^o be occupied by a porous material. Let $\rho(\boldsymbol{r}^o)$ be the relative density of the material ($0 \leq \rho(\boldsymbol{r}^o) \leq 1$) at point $\boldsymbol{P}^o$ of material coordinates $\boldsymbol{r}^o$. For a given distribution of (porous) material, defined by the relative density field $\rho(\boldsymbol{r}^o)$, the goal is to compute the displacements

$$\boldsymbol{u}(\boldsymbol{r}^o, \boldsymbol{\rho}) = \boldsymbol{r}(\boldsymbol{r}^o, \boldsymbol{\rho}) - \boldsymbol{r}^o. \tag{4}$$

Let $d\Omega$ be the volume of a differential region in the vicinity of the point P^o. By definition, the volume occupied by the porous material within the differential region $d\Omega$ is $\rho(\boldsymbol{r}^o)d\Omega$. Therefore, the structural analysis problem with relative density can be written as [17]

$$\begin{aligned}
&\textit{Given} && \rho(\Omega^o) \\
&\textit{find} && \boldsymbol{u} \in \mathcal{H}_u \\
&\textit{such that} && a(\boldsymbol{w},\boldsymbol{u}) = (\boldsymbol{w},\boldsymbol{b})_{\Omega^o} + (\boldsymbol{w},\boldsymbol{t})_{\Gamma^o_\sigma} \;\; \forall \boldsymbol{w} \in \mathcal{H}_w \\
&\textit{being} && a(\boldsymbol{w},\boldsymbol{u}) = \iiint_{\Omega^o} (\boldsymbol{L}\boldsymbol{w})^T \boldsymbol{D}(\boldsymbol{L}\boldsymbol{u})\; \rho\; d\Omega, \\
& && (\boldsymbol{w},\boldsymbol{b})_{\Omega^o} = \iiint_{\Omega^o} \boldsymbol{w}^T\boldsymbol{b}\; \rho\; d\Omega, \\
& && (\boldsymbol{w},\boldsymbol{t})_{\Gamma^o_\sigma} = \iint_{\Gamma^o_\sigma} \boldsymbol{w}^T\boldsymbol{t}\; d\Gamma
\end{aligned} \tag{5}$$

The modifications required to include the effect of the relative density in a standard structural FEM formulation only consist of taking into account the effect of the relative density in the integration. In fact, once the displacements are known, the strain and stress fields are computed with the same expressions (2). The calculations required to compute the stresses and strains do not depend on the material distribution. However, we must exclude the case in which the relative density is locally null, since the concepts of displacement, strain and stress become meaningless.

It is important to remark the physical meaning of the stress $\boldsymbol{\sigma}(\boldsymbol{r}^o)$ computed by means of expressions (2): it represents the homogenized stress tensor of the deforming body. However, we recall that in the vicinity of each point there are probably regions occupied by material, as well as empty ones. Therefore, if we try to analyse the internal balance of forces in a finite subdomain, we should operate with the so-called "effective stress", by multiplying the stress $\boldsymbol{\sigma}(\boldsymbol{r}^o)$ by the relative density $\rho(\boldsymbol{r}^o)$.

2.3 The Finite Element Numerical Model with Relative Density

Let ρ_e be the relative density of element number e, which is assumed constant within the element. Let $\boldsymbol{\rho} = \{\rho_e\}$ $(e = 1, \ldots, N_e)$ be the vector of relative densities (design variables of the topology optimization problem). For a given value of $\boldsymbol{\rho}$, the structural analysis problem requires to obtain the displacements in the discretized space. Thus,

$$\boldsymbol{u}^h(\boldsymbol{r}^o, \boldsymbol{\rho}) = \boldsymbol{u}^p(\boldsymbol{r}^o) + \sum_{i=1}^{N} \boldsymbol{\Phi}_i(\boldsymbol{r}^o)\boldsymbol{\alpha}_i(\boldsymbol{\rho}), \tag{6}$$

where $\boldsymbol{\alpha}_i$ is the vector of nodal displacements of the node i. The functions $\boldsymbol{\Phi}$ are the shape functions used to discretize the geometry of the structure and the displacement field when the element formulation is isoparametric.

Thus, the goal is to

$$\begin{aligned} &\textit{find} && \boldsymbol{\alpha}(\boldsymbol{\rho}) \\ &\textit{such that} && \sum_{i=1}^{N} \boldsymbol{K}_{ji}(\boldsymbol{\rho})\boldsymbol{\alpha}_i(\boldsymbol{\rho}) = \boldsymbol{f}_j(\boldsymbol{\rho}), \quad j = 1, ..., N, \end{aligned} \tag{7}$$

where $\boldsymbol{f}_j(\boldsymbol{\rho})$ is the vector of external forces.

The required terms of the system (7) are given by

$$\begin{aligned} \boldsymbol{K}_{ji}(\boldsymbol{\rho}) &= \sum_{e=1}^{N_e} \boldsymbol{K}^e_{ji}(\rho_e), \\ \boldsymbol{f}_j(\boldsymbol{\rho}) &= \iint_{\Gamma^o_\sigma} \boldsymbol{\Phi}_j^T \boldsymbol{t}\, d\Gamma + \sum_{e=1}^{N_e} \boldsymbol{f}^e_j(\rho_e), \end{aligned} \tag{8}$$

being the elemental contributions

$$\begin{aligned} \boldsymbol{K}^e_{ji}(\rho_e) &= \iiint_{E_e} (\boldsymbol{L}\boldsymbol{\Phi}_j)^T \boldsymbol{D}(\boldsymbol{L}\boldsymbol{\Phi}_i)\ \rho_e\ d\Omega, \\ \boldsymbol{f}^e_j(\rho_e) &= \iiint_{E_e} \left(\boldsymbol{\Phi}_j^T \boldsymbol{b} - (\boldsymbol{L}\boldsymbol{\Phi}_j)^T \boldsymbol{D}(\boldsymbol{L}\boldsymbol{u}^p)\right)\ \rho_e\ d\Omega. \end{aligned} \tag{9}$$

Once the solution $\boldsymbol{\alpha}(\boldsymbol{\rho})$ to problem (7) is found, we can compute at any arbitrary point $\boldsymbol{r}^o \in \Omega^o$ the approximated values of the displacements according to (6). The strains and stresses can be obtained as

$$\boldsymbol{\varepsilon}^h(\boldsymbol{r}^o, \boldsymbol{\rho}) = \boldsymbol{L}\boldsymbol{u}^h(\boldsymbol{r}^o, \boldsymbol{\rho}), \tag{10}$$

$$\boldsymbol{\sigma}^h(\boldsymbol{r}^o, \boldsymbol{\rho}) = \boldsymbol{D}\boldsymbol{\varepsilon}^h(\boldsymbol{r}^o, \boldsymbol{\rho}). \tag{11}$$

Notice that these values are still computed in the usual way although the relative density is being considered. Therefore, if we wish to adapt an existing FEM numerical model of structural analysis as a component of a topology optimization system, we only have to modify the computation of the elemental contributions (9). Moreover, the required adjustment is quite simple, since we only need to introduce the relative density in the integration. Furthermore, this correction is fairly straightforward, since we assume that the relative density is constant within each element. Thus, we only have to multiply the original results by the corresponding relative density of each element.

In addition, the first order sensitivity analysis of the elemental contributions (9) (i.e., the first order derivatives of these contributions with respect to the relative densities of the elements) is immediate. Most of these derivatives are null, since the contribution of each element does not depend on the relative densities of the other elements. Moreover, the first order derivative of each contribution with respect to the relative density of the corresponding element can be easily obtained by just eliminating the relative density in the integration. This is equivalent to evaluate each element contribution type (9) for a value of the relative density $\rho_e = 1$. Consequently, all the second and higher order derivatives of elemental contributions (9) are obviously null. These facts greatly simplify the sensitivity analysis of the problem.

We conclude that with the proposed strategy we do not have to modify the source code at the lower level for adapting an existing FEM code into a topology optimization system. In practice, only slight adjustments must be implemented in the data flow between the higher level routines. In fact, any conventional code should contain all the basic tools to perform the required new computations and the associated sensitivity analysis.

3 Minimum weight with stress constraints formulation

3.1 Optimization Problem

The optimization problem can be formulated from a generic point of view as

$$\begin{array}{lll} \text{Minimize} & F(\boldsymbol{\rho}) = Cost(\boldsymbol{\rho}) & \\ \text{subject to:} & g_j(\boldsymbol{\rho}) \leq 0 & j = 1, ..., m \\ & 0 < \rho_{min} \leq \rho_e \leq 1, & e = 1, ..., N_e \\ & \rho_{min} = 0.001 & (usually) \end{array} \tag{12}$$

where $F(\boldsymbol{\rho})$ is the objective function and $g_j(\boldsymbol{\rho})$ are the stress constraints. The minimum value of the relative density ($\rho_{min} = 0.001$) is the most common value in the bibliography [2,3]. Other values of ρ_{min} could be considered but they give similar results.

3.2 Statement of the objective function

The objective function can be defined as

$$F(\boldsymbol{\rho}) = \sum_{i=1}^{N_e} \int_{\Omega_i} (\rho_i)^{1/p} \gamma_{mat} \, d\Omega \tag{13}$$

where the parameter p is a penalty parameter which aim is to avoid intermediate densities in the optimized solution [23,24]. If no penalization is used ($p = 1$) the objective function to be minimized is the total weight of the structure. This is the most common situation, but some examples with different penalization parameter will be also presented in this paper to force 0-1 solutions. Physically, a value of the parameter less than 1 gives preference to the intermediate densities. A value of the parameter higher than 1 penalizes the intermediate densities.

It is very important to remark that the problem stated according to (12) is highly non-linear. In addition, it is also non-convex even without using any penalization of the intermediate densities. However, as it can be observed in the application examples, the solutions obtained with this formulation are very appropriate and realistic.

3.3 Statement of the Stress Constraints

The statement of the stress constraints is based on the values $\boldsymbol{\sigma}^h(\boldsymbol{r}^o, \boldsymbol{\rho})$ computed by means of (6), (10) and (11). These values are used in the local stress constraints approach and in the global stress constraint approach in order to define optimization problems with suitable and feasible structural solutions. We present these two different approaches and discuss their advantages and disadvantages.

3.3.1 Local stress constraints

The value of the local constraints approach is computed by limiting the maximum value of the local stress obtained by means of the Finite Element formulation (6), (10) and (11). Moreover, the material failure is usually checked according to stress failure criteria (e.g. Von Mises criterion). These criteria usually consider a "reference stress", based on the homogenized stress tensor, to test the failure of the material. Thus, the maximum allowable values of this "reference stress" $\widehat{\sigma}(\boldsymbol{\sigma}^h)$ at each point $\boldsymbol{r}_j^o$ can be considered by introducing the following inequalities:

$$\begin{array}{l} g_j(\boldsymbol{\rho}) = \widehat{\sigma}\Big(\boldsymbol{\sigma}^h(\boldsymbol{r}_j^o, \boldsymbol{\rho})\Big) - \widehat{\sigma}_{max} \leq 0, \quad or \\ g_j(\boldsymbol{\rho}) = \widehat{\sigma}_{min} - \widehat{\sigma}\Big(\boldsymbol{\sigma}^h(\boldsymbol{r}_j^o, \boldsymbol{\rho})\Big) \leq 0, \end{array} \tag{14}$$

where $\widehat{\sigma}_{max}$ and $\widehat{\sigma}_{min}$ are the corresponding upper and lower limits of the stress failure criterion used.

Consequently, this approach requires to manage a huge number of highly non-linear constraints, which deals with a very complicated optimization problem. We have solved this optimization problem by using a Sequential Linear Programming algorithm (based on the Simplex Method) to obtain the search direction and a Quadratic Line Search algorithm to obtain the advance factor [19,21]. Thus, all the first order derivatives of the objective function and of the constraints must be computed in order to obtain the search direction. In addition, the first and the second order directional derivatives of the objective function and of the constraints must be computed in order to obtain the advance factor.

The required Sensitivity Analysis is developed following the general formulation proposed in [20]. The full set of first order derivatives of the stress constraints are computed by means of the adjoint state method, while the first and second order directional derivatives are computed by means of the direct differentiation method. In both cases, the implemented procedures are exact from the analytical point of view (i.e., no finite difference approximations are performed). Furthermore, both the computing time and the required amount of data storage are minimized. In practice, most of the computing effort and the amount of data storage involved in the sensitivity analysis are devoted to computing the full set of first order derivatives of the stress constraints.

On the other hand, the stress constraints approaches usually present the so called "singularity phenomena". The optimum solution is a singular point, from a theoretical point of view, of the feasible solutions [5,11]. Cheng and Jiang [4] explained the nature of this phenomenon, which is due to the discontinuous nature of

the stresses when the density tends to zero. The stress constraints become meaningless when the relative density is exactly null. However, the stress constraints may become violated when the relative density tends to zero. Numerical optimization algorithms are not able to detect this fact and, consequently, the optimal solution may not be reached. They do not remove all the material (although this may be the optimum solution) because the stress constraint may become violated as the relative density gets closer to zero. Thus, the stress constraints present a discontinuity at zero density.

This fact can be observed in some theoretical truss optimization problems such as that proposed by Cheng and Guo [5,12]. Furthermore, it has been demonstrated that singularity phenomena must be also considered in other fields of structural optimization [6,23].

For example, this phenomenon happens when one section of a structure subjected to the maximum bending moment increases its height by adding porous material in the maximum stress edges (figure 1). When the relative density of this new added material is decreased the stress becomes higher than the maximum allowable. However, when the material is completely removed the solution becomes feasible again because the constraint is removed. The optimum is to remove all the added material but the numerical algorithm is not able to detect this singularity [23].

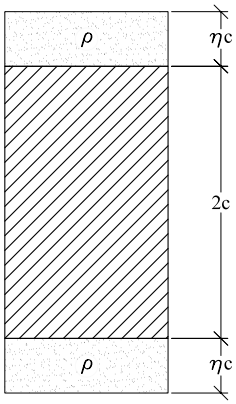

Fig. 1 Rectangular beam subjected to horizontal bending moment [23].

Consequently, the formulation needs to be relaxed in order to avoid this undesiderable situation. In this paper, we propose a different formulation based on the contributions of Duysinx and Cheng [5,6] and Navarrina [22]. Following these ideas, we propose the following statement of the local stress constraints:

$$\begin{aligned} g_j(\boldsymbol{\rho}) &= \left[\widehat{\sigma}\left(\boldsymbol{\sigma}^h(\boldsymbol{r}_j^o,\boldsymbol{\rho})\right) - \widehat{\sigma}_{max}\right]\ \rho(\boldsymbol{r}_j^o)^q \leq 0, \quad or \\ g_j(\boldsymbol{\rho}) &= \left[\widehat{\sigma}_{min} - \widehat{\sigma}\left(\boldsymbol{\sigma}^h(\boldsymbol{r}_j^o,\boldsymbol{\rho})\right)\right]\ \rho(\boldsymbol{r}_j^o)^q \leq 0, \end{aligned} \tag{15}$$

where q is an exponent that takes the value $q = 0$ if constraints are imposed on the homogenized stress tensor or $q = 1$ if constraints are imposed on the effective stress tensor.

If the material being used presents an equal response under tensile and compressive forces (e.g. steel) the Von Mises failure criterion can be used. Thus, the stress constraints (15) can be reduced to only one stress constraint. This stress constraint can be formulated as

$$g_j(\boldsymbol{\rho}) = \left[\widehat{\sigma}\left(\boldsymbol{\sigma}^h(\boldsymbol{r}_j^o,\boldsymbol{\rho})\right) - \widehat{\sigma}_{max}\right]\ \rho(\boldsymbol{r}_j^o)^q \leq 0. \tag{16}$$

A more general formulation to impose local stress constraints which avoids singularity phenomena can be obtained by relaxing the formulation according to [6]. Thus,

$$g_j(\boldsymbol{\rho}) = \left[\widehat{\sigma}\left(\boldsymbol{\sigma}^h(\boldsymbol{r}_j^o,\boldsymbol{\rho})\right) - \widehat{\sigma}_{max}\ \varphi_j\right]\ \rho(\boldsymbol{r}_j^o)^q \leq 0, \tag{17}$$

where φ_j is the "stress relaxation coefficient" obtained by:

$$\varphi_j = 1 - \varepsilon + \frac{\varepsilon}{\rho(\boldsymbol{r}_j^o)}. \tag{18}$$

The "relaxation parameter" ε usually varies from 0.001 to 0.1, and its value is reduced when the solution gets closer the optimum during the optimization process. The effect of the "stress relaxation coefficient" can be observed in figure 2. As we have mentioned before, the

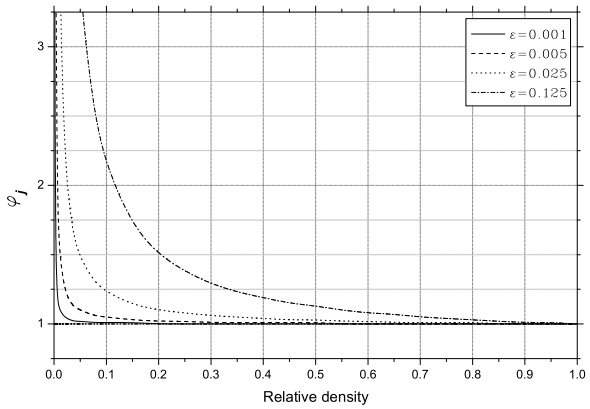

Fig. 2 Stress relaxation coefficient (φ_j)

exponent q permits the multiplication of the constraint by the relative density in order to avoid some singularities. This topic has been extensively analysed by the authors and it can be found in [23].

On the other hand, the ε-relaxation introduces some unexpected effects in the formulation. Thus, Stolpe and Svanberg explained that the trajectories to the optimum produced by the relaxed approach are not unique and they depend on the initial design [31]. This fact makes the problem non-convex and much more complicated. However, the solutions obtained with this formulation are suitable if appropriate values of the "relaxation parameter" ε are used.

Now we can study this singularity phenomenon by means of the relaxed formulation proposed in (17) and (18).

Figures 3 and 4 show the effect of using an exponent $q = 0$ and $q = 1$ respectively. These figures have been obtained by using a parameter $\eta = 0.1$ (see figure 1). In addition, a "stress relaxation coefficient" (φ_j) is also considered to test different possibilities of the best value of these coefficients.

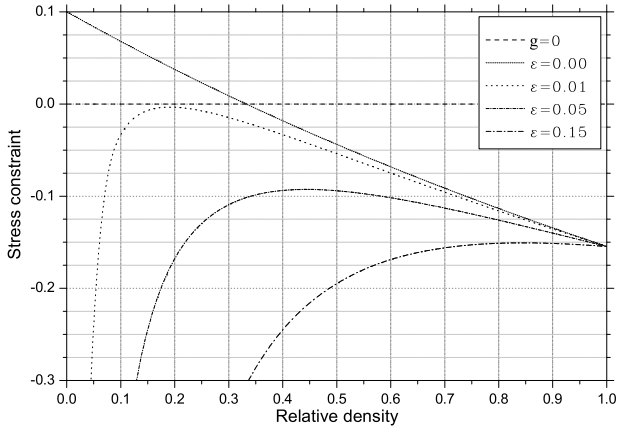

Fig. 3 Stress constraint (17) with $q = 0$.

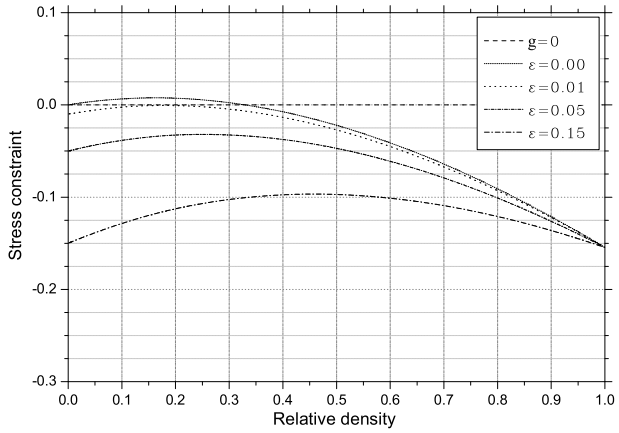

Fig. 4 Stress constraint (17) with $q = 1$.

The two possibilities should work properly and usually produce similar solutions. However, in some specific situations, we have observed that the exponent $q = 1$ produces better results.

3.3.2 Global stress constraint

The global stress constraint approach is a relatively recent field in topology optimization of continuum structures. This approach implies the substitution of all the local constraints by only one constraint that includes all of them. This approach presents obvious advantages since the optimization problem is much easier to solve compared with the local approach because only one constraint has to be considered. In addition, the calculus of the sensitivity analysis requires much smaller computational effort. Thus, the data storage amount and the computing time are decreased.

The keystone of this approach is the global function that aggregates all the local constraints into one single global constraint. In this paper we propose a global formulation based on the Kreisselmeier-Steinhauser function as it was used by Martins and Poon [16] in aerostructural optimization. However, we have introduced some simple modifications to solve a number of numerical effects observed from the original one. Thus, the proposed global function of constraints aggregation can be formulated as

$$G_{KS}\left(\boldsymbol{\rho}\right) = \frac{1}{\mu}\, ln\left(\sum_{j=1}^{N_e} e^{\;\mu\left(\frac{\widehat{\sigma}_j \;-\; \widehat{\sigma}_{max}}{\widehat{\sigma}_{max}}\right)}\right) \tag{19}$$

where $\widehat{\sigma}_j = \widehat{\sigma}_j\left(\boldsymbol{\sigma}^h(\boldsymbol{r}_o^j, \boldsymbol{\rho})\right)$ are the local stresses considered in the previous formulation and $\widehat{\sigma}_{max}$ is the maximum allowable stress according to the failure criterion considered. Thus, the parameter μ does not depend on the stress units and the exponent of the global function becomes dimensionless. In addition, this modification also avoids the numerical overflow because the exponent does not take too large values when the local constraints become violated.

The parameter μ is just a tuning coefficient which intends to penalize the failure to satisfy the local constraints. In theory, global function (19) tends to the value $\max_j\{(\widehat{\sigma}_j - \widehat{\sigma}_{max})/\widehat{\sigma}_{max}\}$ as the parameter μ tends to infinity. This is the key idea that allows us to aggregate all the local constraints into one single global constraint. Therefore, the global constraint will not adequately represent the corresponding whole set of local constraints if the value of μ is not large enough. However, when the value of μ is too large the problem becomes unstable due to the increasing non-linearity of the global function and the intrinsic numerical inaccuracy of the involved computations. In addition, some

undesirable computational conditions (such as overflow) are more likely to occur.

The maximum value of the global function can be obtained by taking the maximum value of all the local constraints ($\widehat{\sigma}_j - \widehat{\sigma}_{max} = 0, \quad j = 1, ..., N_e$). Thus,

$$G_{KS_{max}} = \frac{1}{\mu}\, ln(N_e), \tag{20}$$

where N_e is the total number of local stress constraints considered. Then, the global constraint that substitutes all the local constraints type (16) can be expressed as

$$G_{KS}\left(\boldsymbol{\rho}\right) - G_{KS_{max}} \leq 0, \tag{21}$$

which is equivalent (in theory) to the whole set of local constraints type (16) as the parameter μ tends to infinity. Thus, for large enough values of the parameter μ, global constraint (21) should perform adequately by replacing the whole set of local constraints.

In addition, this formulation can be corrected to avoid the singularity phenomena explained in the previous section. Furthermore, only simple modifications in the calculus of the global function and in the sensitivity analysis should be carried out. Thus, the singularity phenomena in the global stress function can be avoided by redefining (19) as follows:

$$G_{KS}\left(\boldsymbol{\rho}\right) = \frac{1}{\mu}\, ln\left(\sum_{j=1}^{N_e} e^{\,\mu\left(\frac{\widehat{\sigma}_j}{\widehat{\sigma}_{max}\,\varphi_j} - 1\right)}\right) \tag{22}$$

where φ_j has been previously defined in (18). The global formulation is based on the relaxation of the local constraints contributions to the aggregation function. Thus, the relaxation of the global function has been taken into account by using the relaxed approach of the local stress constraints proposed.

Now, problem (12) is stated as a constrained optimization problem with a single highly non-linear constraint. The objective function is the weight or the cost of the structure (13). In addition, a penalization factor such as that proposed in (13) is also considered according to the value of the intermediate density penalization parameter.

Consequently, we have turned our attention to "barrier function" type optimization algorithms. We have tested different types of "barrier functions" (logarithmic and inverse) obtaining similar results.

The resulting problem can be solved by using unconstrained optimization algorithms. However, it is very important to take into account the non-linearity of the global function and, of course, the non-linearity of the "inverse barrier function" (ϕ). Due to this fact, the most appropriate technique to solve this problem would be a second order Newton or Quasi-Newton type method. However, these techniques require to store an approximation to the Hessian matrix. In practice, this approximation is not advisable in topology optimization due to the large number of design variables involved.

The iterative modification of the solution is obtained in two steps. The first step finds the direction of modification of the design. The second step searches the advance factor in the search direction previously obtained.

We use the conjugate gradient method proposed by Fletcher-Reeves [9] in order to obtain the search direction because it only requires to store a small number of values.

Once the search direction has been obtained, the advance factor is computed by using second order directional derivatives of the objective function and that of the constraints (such as that proposed in [20]) in order to avoid the "zig-zag" phenomenon near the optimum solution.

4 Numerical examples

We present some examples with different structural configurations. The proposed examples are solved with the local constraints approach and the global constraints approach. Then, the results obtained with both formulations are compared.

The proposed examples herein are two dimensional structures solved in plane stress under different load conditions and support configurations. However, we draw three dimensional structures by assuming that the relative density is the thickness of each element. This is not strictly true because the physical meaning of the relative density is not the thickness. The relative density means the material state of each element of the mesh. However, we represent the solution with this assumption because it is easier to visualize a three dimensional figure than a two dimensional one.

4.1 Local constraints formulation

We present three examples solved with the local constraints formulation. Each one of these examples has been solved with different values of the coefficients (ε, q and p) presented in the formulation to show their effect. The problems proposed are two-dimensional structures solved in plane stress with 1 m of thickness.

4.1.1 L-shape beam

The first example is a L-shape beam 1 m long and 1 m high (figure 5). This beam is supported along the

upper edge. A vertical force of 4 10^3 kN is applied at the mid point of the right vertical edge distributed on two elements. The thickness of the structure is 1 m. In addition, self weight is also considered.

The beam is made of steel with an elastic limit of σ_e=230 MPa and a Young Module of $E_e = 2.1\ 10^5$ MPa. The Poisson value is $\nu = 0.3$ and the mass density is $\gamma_{mat} = 76.5$ kN/m^3.

The finite element mesh consists of 1024 quadrilateral elements.

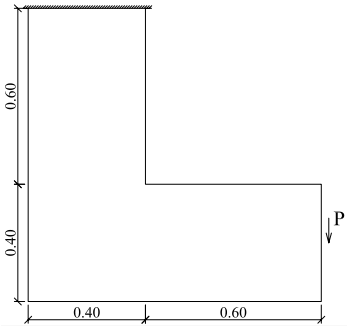

Fig. 5 L-shape beam (units in meters).

Figure 6 shows the solution of the L-shape beam without any penalization of the intermediate densities. As can be observed, extensive areas with intermediate densities appear without penalization. However, if the intermediate densities are penalized, the final design tends to solid-void configurations. This fact can be observed in figure 7.

This solution (figure 7) is very similar to the solutions obtained by Duysinx and Bendsøe [6] or Pereira et Al. [25]. In addition, it is also similar to the analytical solution obtained by Lewinsky *et al.* [15] for L-shape beams. It is important to remark that the total weight of the solutions obtained by penalizing intermediate densities (figure 7) is slightly higher than the total weight of the solution without penalization (figure 6) because of the dependency of the objective function on the penalization parameter.

4.1.2 Large beam with small height

The second example (figure 9) is a 40 m long and 1 m high beam. The horizontal deformation is not restricted. A vertical force of 10^4 kN is applied at the center of the left span distributed onto the upper edges of two contiguous elements. The material has the same properties as in example 1 and self weight is also considered. The finite element mesh consists of $72 \times 15 = 1080$ elements and the thickness is 1 m. In addition, no penalization of the intermediate densities has been used.

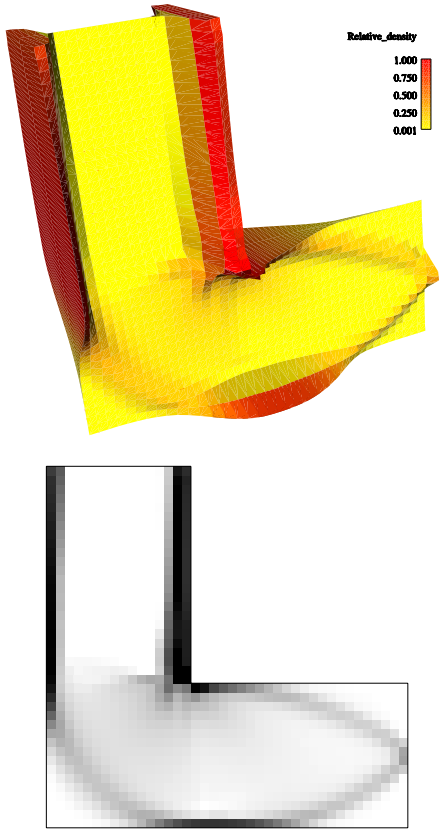

Fig. 6 Local approach with q=1, $\varepsilon = 0.01$ and $p = 1$ (F=18.0% F_o)

This example is very interesting because the solutions presented in figure 10 without any relaxation ($\varepsilon = 0$) and figure 11 with $\varepsilon = 0.005$ do not agree with the solutions obtained by means of maximum stiffness formulations. This fact can be easily checked by solving the same problem with the interactive application of the TOPOPT homepage [30]

The solutions obtained with maximum stiffness approaches change significantly with the value of the constraint (the maximum quantity of material to be used). If the maximum quantity of material to be used is smaller than 50 % of the material available in the domain, the maximum stiffness solution corresponds to a material distribution in the left span. In this situation the left span is full of material, while the right one is void. However, this solution is unfeasible because in some points the material failure stress is exceeded according to the loads applied and the material properties.

Furthermore, if the maximum quantity of material to be distributed is greater than 50 % of the maximum possible in the domain of the structure, the maximum stiffness solution corresponds to a material distribution in the two spans. In this situation, the left span (50 % of

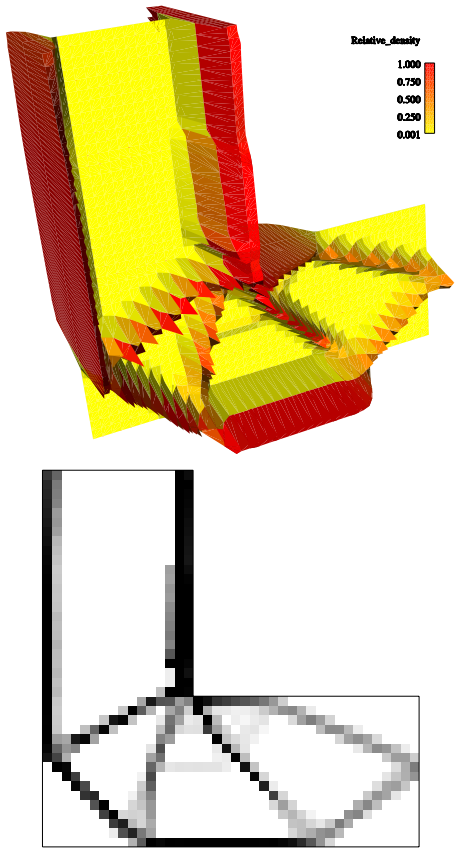

Fig. 7 Local approach with q=1, $\varepsilon = 0.01$ and $p = 4$ (F=19.2% F_o)

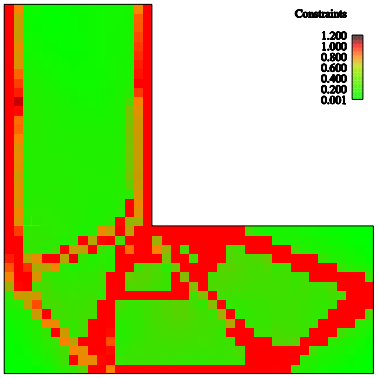

Fig. 8 Normalized stress state of the L-shape beam solution with the local approach ($p = 4$, $q = 1$, $\varepsilon = 0.01$)

the maximum volume of the structure) is almost full of material and the rest is distributed into the right one.

Thus, in this formulation the value of the material constraint influences considerably on the final topology of the structure although the applied loads do not change. The solution obtained with the minimum weight formulation with stress constraints presents a material distribution in both spans to avoid the structural failure. If no material is distributed in the right span, the

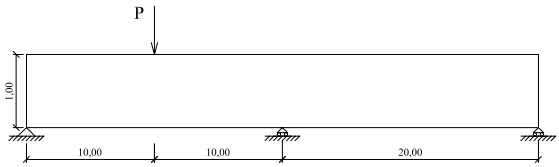

Fig. 9 Large beam with small height (units in meters)

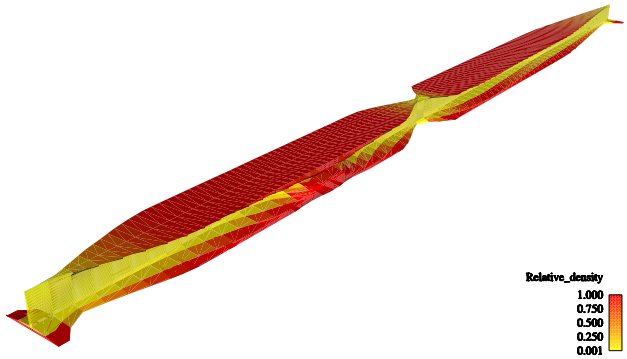

Fig. 10 Local approach solution with q=1, $\varepsilon = 0.000$, $p = 1$ (F=23.5% F_o)

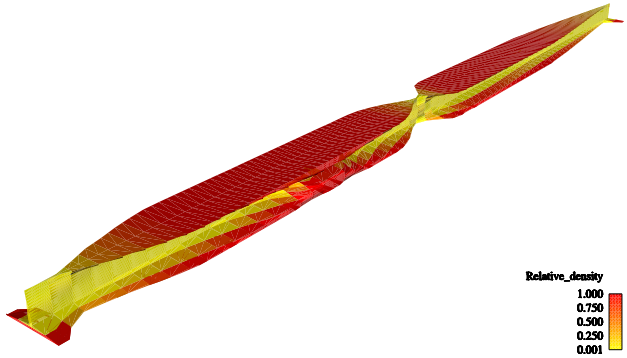

Fig. 11 Local approach solution with q=1, $\varepsilon = 0.005$, $p = 1$ (F=22.8% F_o)

maximum stress does not satisfy the material failure criterion and, consequently, the solution is not feasible. Thus, the minimum weight with stress constraints formulation produces adequate material distributions and guarantees the feasibility of the solution. In addition, this solution varies with the applied loads and the material failure stress, which is more realistic than a volume constraint.

4.1.3 Beam with large height

The third example is a 40 m long and 15 m high beam (figure 12). The external load is a force of 6 10^5 kN distributed onto the upper edges of two contiguous elements and applied at a distance of 1/3 of the total length from the left support. In addition, self weight is also considered. The material has the same properties as in example 1. We have used a mesh with $30 \times 15 =$

450 rectangular elements. In addition, no penalization of the intermediate densities has been used.

Figure 13 shows the solution to this example by using an exponent q=1 and without any relaxation ($\varepsilon = 0$). Figure 14 shows the solution to this example with a relaxation parameter $\varepsilon = 0.005$ and $q = 1$.

As can be seen, the results obtained using our approach (MWSC) are the expected ones from an engineering point of view since they are very similar to an arch (figures 13 and 14).

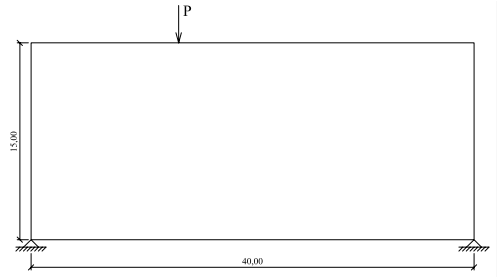

Fig. 12 Beam with large height (units in meters)

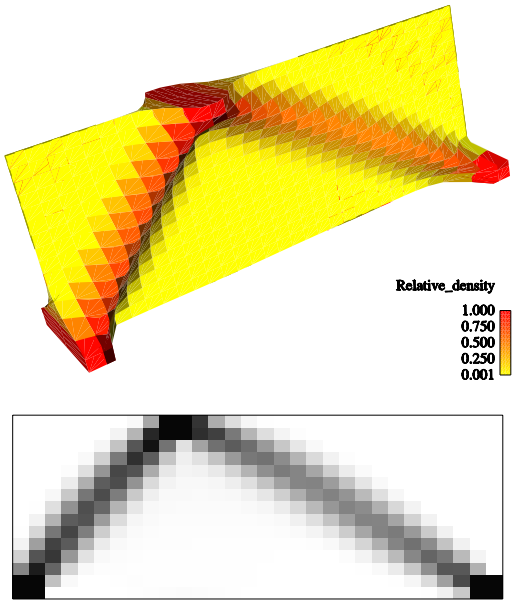

Fig. 13 Local approach solution with q=1, $\varepsilon = 0.000$, $p = 1$ (F=16.7% F_o)

This example shows a high reduction of the objective function because most of the elements of the mesh are void at the end of the optimization process. If we take into account this fact we can refine the mesh by dividing each element into four new elements and eliminating all the voids. With this refining process, it is possible to compute the same problem by using a small

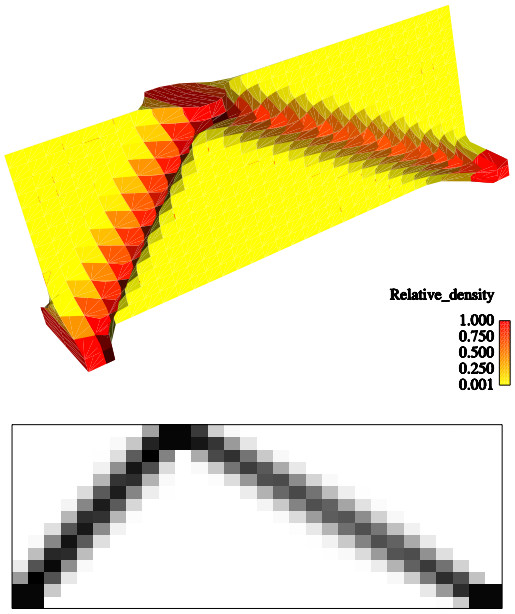

Fig. 14 Local approach solution with q=1, $\varepsilon = 0.005$, $p = 1$ (F=16.1% F_o)

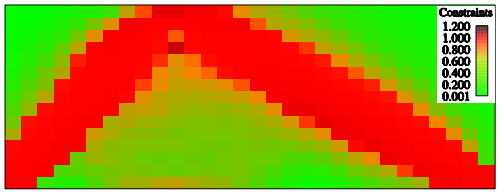

Fig. 15 Normalized stress state of the beam with large height solution with the local approach ($q = 1$, $p = 1$, $\varepsilon = 0.005$)

number of elements and constraints. Consequently, the new initial solution is shown in figure 16 and the new material distribution is shown in figure 17. It is important to remark the huge reduction of computing effort since only 808 elements are required to solve the problem. However, if the void elements are not removed, 1800 elements and constraints are required.

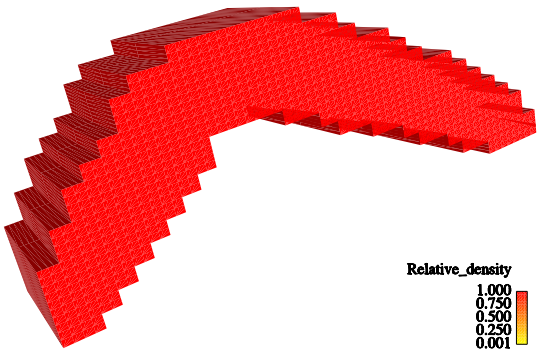

Fig. 16 Initial solution of the refined mesh (q=1)

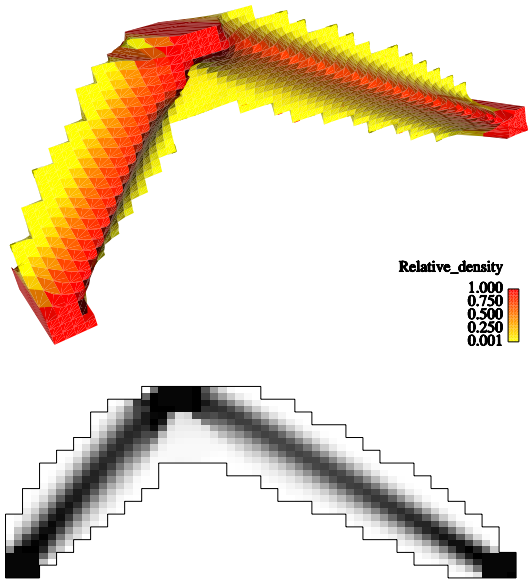

Fig. 17 Local approach solution of the refined mesh with q=1, $\varepsilon = 0.005$ (F=17.0% F_o)

In table 1 the stress state is analysed via the value of the most violated constraint of each example. It can be observed that, in general, the stress constraints are not violated and the most violated constraint is only slightly exceeded. This fact is very important because it guarantees the feasibility of the obtained solution. In figures 8 and 15, the stress state of all the elements can be checked. We have used normalized stresses by dividing each stress by the maximum allowable stress in order to simplify the pictures.

Table 1 Summary of the local approach examples

	q	ε	p	N_{iter}	$\frac{F}{F_o}$	$max\left\{\frac{G_j}{\widetilde{\sigma}_j}\right\}$
Fig. 6	1	0.010	1	130	18.00 %	2.32 %
Fig. 7	1	0.010	4	150	19.20 %	1.45 %
Fig. 10	1	0.000	1	150	23.50 %	1.00 %
Fig. 11	1	0.005	1	150	22.80 %	0.64 %
Fig. 13	1	0.000	1	100	16.70 %	6.48 %
Fig. 14	1	0.005	1	150	16.10 %	1.85 %
Fig. 17	1	0.005	1	111	17.00 %	4.30 %

4.2 Global constraint formulation

In this section, we show some of the previous examples solved with the global approach to compare the obtained results with both formulations (the local and the global). The global constraint formulation aggregates all the local constraints with a global function that penalizes the most violated constraints in order to make them feasible. Thus, the global constraints formulation is less strict than the local formulation because it does not guarantee the feasibility of the solution from the local constraints point of view. In addition, the local constraints formulation requires less tuning parameters.

In general, the solutions obtained with both formulations (local and global) are very similar, although they are not identical, due to the aggregation of the stress constraints.

However, when the number of design variables (and constraints) increases, the computing time of the local approach becomes prohibitive. On the other hand, the computing time and the data storage amount required by the global approach are much smaller.

Thus, it is more appropriate to use the global constraints approach when the number of elements (and constraints) is very large. The computing effort required by the local approach becomes prohibitive and even unaffordable when the refinement of the mesh is increased. On the other hand, the local approach imposes a more strict control over stress constraints and gives better results. Consequently, the refinement technique proposed in the third example of the local constraints approach is a very good solution to reduce the size of the problem without loosing precision. This size reduction allows us to decrease much of the computing time and the amount of data storage when the local constraints approach is used.

4.2.1 L-shape beam

This example solves the same problem proposed in figure 5. We use the material properties and the dimensions proposed in the local approach solution. If we compare the solutions obtained by means of the local approach (figures 6 and 7) and the solutions obtained by means of the global approach (figures 18 and 20), the results are very similar with insignificant differences. The final topology of the structure is essentially equivalent. In addition, the weight reduction is also similar.

4.3 Large beam with small height

This example solves the large beam with small height proposed in figure 9. Thus, we use the geometry, the material properties and the loads proposed in the local approach.

The solutions obtained by means of the global constraint formulation (figures 22 and 23) and the local constraints formulation (figures 10 and 11) are very similar. However, the weight reduction obtained with the

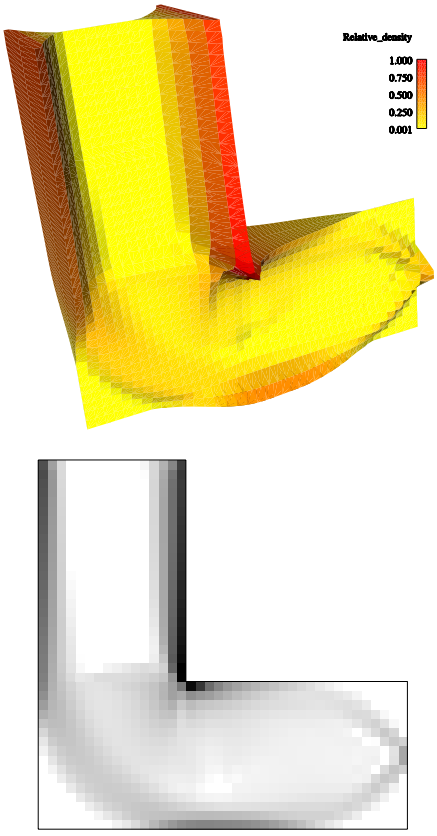

Fig. 18 Global approach with μ=20, p=1, ε=0.01 (F=15.9% F_o)

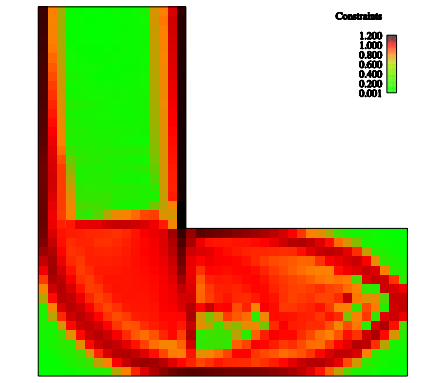

Fig. 19 Normalized stress state of the L-shape beam solution with the global approach (p=1, ε=0.01)

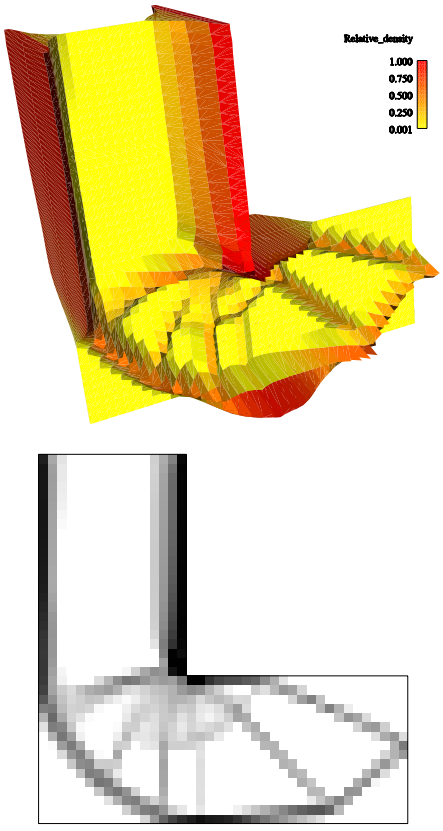

Fig. 20 Global approach with μ=20, p=4, ε=0.01 (F=17,6% F_o)

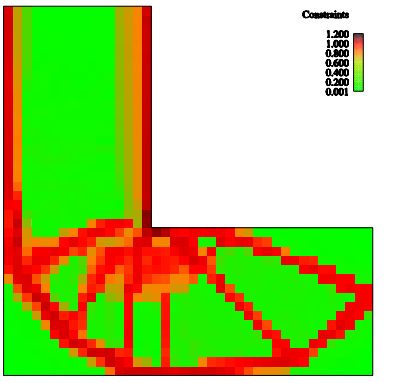

Fig. 21 Normalized stress state of the L-shape beam solution with the global approach (p=4, ε=0.01)

global approach is slightly greater than that obtained with the local approach due to the aggregation of the constraints.

4.3.1 Beam with large height

This example solves the beam with large height proposed in figure 12. Thus, we use the geometry, the material properties and the applied loads proposed in the local approach. The Finite Element mesh has $30 \times 15 = 450$ rectangular elements.

The solutions obtained by means of the global constraint formulation (figures 24 and 26) and the local constraints formulation (figures 13 and 14) are similar. The structural topology is equivalent but the global approach does not strictly satisfy the local stress constraints. The weight reduction is also similar for both formulations, but it is slightly greater in the global approach solution due to the aggregation of the constraints.

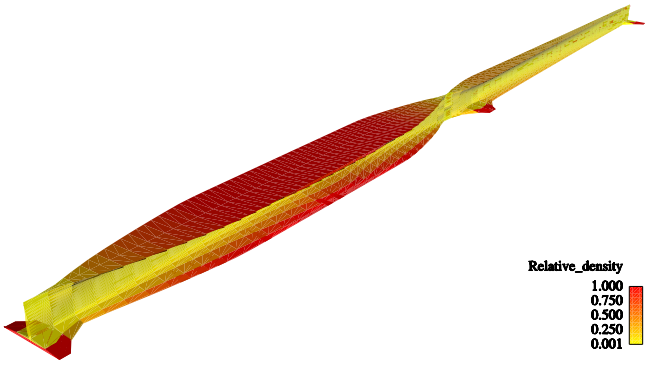

Fig. 22 Global approach with μ=20, p=1, ε=0.02 (F=17.0% F_o)

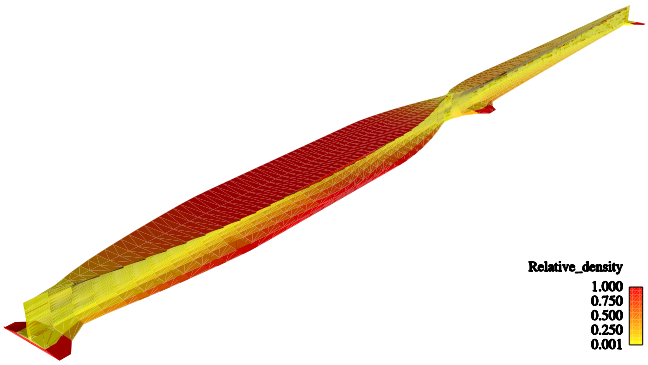

Fig. 23 Global approach with μ=20, p=4, ε=0.02 (F=17.5% F_o)

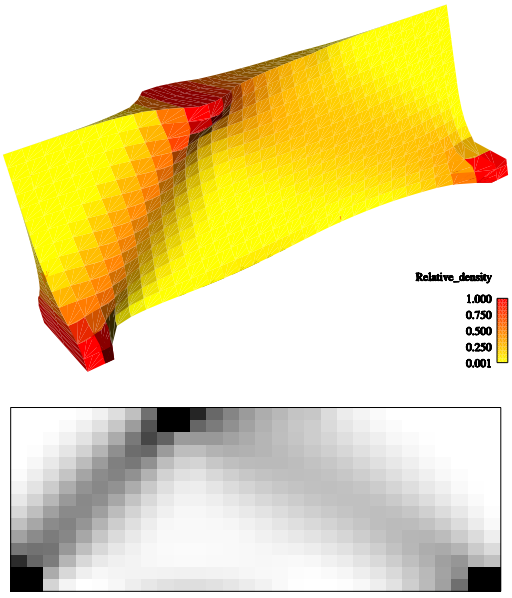

Fig. 24 Global approach with μ=10, p=1, ε=0.005 (F=18,0% F_o)

Table 2 shows the results obtained with the global approach for different values of the most important parameters. In addition, it is possible to compare the numerical results of the two approaches, local and global, by comparing the results of the tables 1 and 2. As can be observed in table 2, the local stress constraints are more

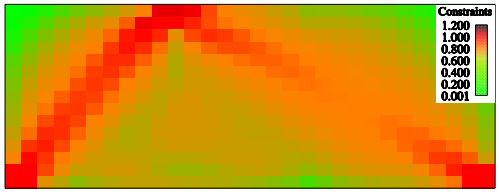

Fig. 25 Normalized stress state of the beam with large height solution with the global approach ($p = 1$, ε=0.005)

violated when the global constraints approach is used. This fact is easily explained due to the aggregation of all the local constraints and the corresponding loss of accuracy. However, the obtained results seem to be correct because the topologies obtained via the global constraints approach are quite similar to the ones obtained via the local approach. In addition, the stress state can improve if the parameter μ is increased at the end of the optimization process. This modification makes the problem even more non-linear but it also makes the solution more feasible from a local stress constraints point of view.

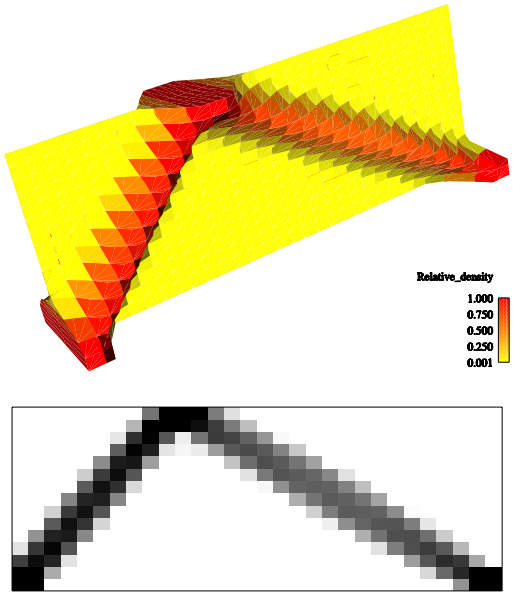

Fig. 26 Global approach with μ=10, p=4, ε=0.005 (F=18,3% F_o)

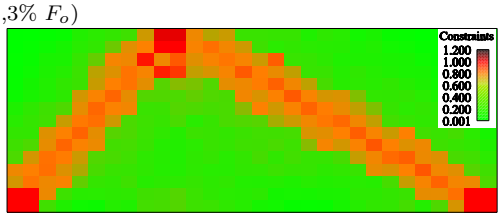

Fig. 27 Normalized stress state of the beam with large height solution with the global approach ($p = 4$, $\varepsilon = 0.005$)

Table 2 Summary of the global approach examples

	μ	ε	p	N_{iter}	$\frac{F}{F_o}$	$max\left\{\frac{G_j}{\widetilde{\sigma}_j}\right\}$
Fig. 18	20	0.010	1	351	15.90 %	13.98 %
Fig. 20	20	0.010	4	593	17.60 %	9.22 %
Fig. 22	20	0.020	1	453	17.00 %	13.35 %
Fig. 23	20	0.020	4	345	17.50 %	11.61 %
Fig. 24	10	0.005	1	200	20.29 %	1.13 %
Fig. 26	10	0.005	4	200	18.30 %	0.52 %

5 Conclusions

We have presented two different approaches to solve minimum weight structural topology optimization problems with stress constraints: the local constraints approach and the global constraints approach.

In both cases, the structural analysis formulation is based on a conventional FEM approach with simple modifications.

In the local constraints approach, a constraint is imposed at a given point within each element. This gives rise to an optimization problem involving a very large number of highly non-linear constraints. In addition, a relaxation technique is presented in an attempt to avoid the singularity phenomena.

The global constraints approach is based on the function proposed by Kreisselmeier and Steinhauser [16]. This function aggregates all the local constraints and highly penalizes the most violated ones. In addition, this formulation permits the consideration of a relaxed version of the constraint in a similar way as we proposed for the local constraints approach.

The presented optimization approach does not require stabilization techniques. When the local formulation is used, no artificial parameters are required. In the global constraints approach, it is necessary to use some artificial coefficients with a clear physical interpretation. Thus, it is very easy to understand the meaning of these coefficients and their best values to use. Intermediate densities are not usually penalized. However, when the intermediate densities are penalized, the solutions become truss like structures as it was expected.

The objective function and the constraints have a clear physical interpretation from an engineering point of view. In addition, another kind of constraints can be used (displacements, vibration frequencies) and several load cases can be analysed simultaneously with this formulation.

The proposed formulation is very robust, especially when the local stress constraints approach is considered. However, this approach imposes a large number of non-linear constraints and, consequently, requires large data storage and large computing time. The global approach loses some control over the local constraints. However, it allows us to consider more refined meshes with a large number of elements.

Acknowledgements This work has been partially supported by Grant Numbers DPI-2006–15275, DPI-2007–61214 of the *"Ministerio de Ciencia y Tecnología"* of the Spanish Government, by Grant Numbers PGIDIT03–PXIC118001PN and PGIDIT03–PXIC118002PN of the *"Dirección Xeral de I+D"* of the *"Consellería de Innovación, Industria e Comercio"* of the *"Xunta de Galicia"*, and by research fellowships of the *"Universidade da Coruña"* and the *"Fundación de la Ingeniería Civil de Galicia"*.

References

1. Bendsøe M. P. and Kikuchi N. (1988) Generating optimal topologies in structural design using a homogenization method. Comp. Methods in Applied Mechanics and Eng., Vol. 71, 197–224.
2. Bendsøe M. P. (1989) Optimal shape design as a material distribution problem. Structural Optimization, Vol. 1, 193–202.
3. Bendsøe M. P. (1995) Optimization of structural topology, shape, and material. Springer-Verlag, Heidelberg.
4. Cheng G. and Jiang Z. (1992) Study on topology optimization with stress constraints. Eng. Optimization, Vol. 20, 129–148.
5. Cheng G. D. and Guo X. (1997) ε-relaxed approach in structural topology optimization. Structural Optimization, Vol. 13, 258–266.
6. Duysinx P. and Bendsøe M. P. (1998) Topology optimization of continuum structures with local stress constraints. Int. Journal for Numerical Methods in Engineering, Vol. 43, 1453–1478.
7. Duysinx P. (1998) Topology optimization with different stress limits in tension and compression. International report: Robotics and Automation, Institute of Mechanics, University of Liege, Liege, Belgium.
8. Duysinx P. and Sigmund O. (1998) New Developments in Handling Stress Constraints in Optimal Material Distributions. 7th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Design Optimization, Saint Louis, Missouri, USA.
9. Fletcher R. (1987) Practical Methods of Optimization. Wiley, Scotland, UK.
10. Hughes T. J. R. (1987) The finite element method: linear static and dynamic Finite Element analysis. Prentice-Hall, Englewood Cliffs.
11. Kirsch U. (1990) On singular topologies in optimum structural design. Structural Optimization, Vol. 2, 133–142.
12. Kirsch U. and Topping B. H. V. (1992) Minimum Weight Design of Structural Topologies. Journal of Structural Engineering, ASCE, Vol. 118, 1770–1785.
13. Kohn R. V. and Strang G. (1986) Optimal design and relaxation of variational problems. Communications on Pure and Applied Mathematics, Vol. 39, 1–25 (Part I), 139–182 (Part II), 353–377 (Part III).
14. Liang Q.Q. et al. (1999) Optimal selection of topologies for the minimum-weight design of continuum structures with stress constraints. Journal of Mechanical Engineering Science, Proceedings of The Institution of Mechanical Engineers, Part C, 213(8), 755-762, UK.
15. Lewinsky T., Rozvany G. I. N. (2008) Exact analytical solutions for some popular benchmark problems in topology optimization III: L-Shaped domains. Structural and Multidisciplinary Optimization, Vol. 35, 165–174.

16. Martins J. R. R. A. and Poon N. M. K. (2005) On Structural Optimization Using Constraint Aggregation, (WCSMO6) Proc. VI World Congress on Structural and Multidisciplinary Optimization, Rio de Janeiro, Brazil.
17. Muíños I. (2001) Optimización Topológica de Estructuras: Una Formulación de Elementos Finitos para la Minimización del Peso con Restricciones en Tensión (in Spanish), Technical Report, ETSICCP, Universidade da Coruña, A Coruña.
18. Muíños I. et al. (2002) Una formulación de mínimo peso con restricciones en tensión para la optimización topológica de estructuras (in Spanish). Métodos Numéricos en Ingeniería y Ciencias Aplicadas, CIMNE, 399–408, Barcelona.
19. Navarrina F. and Casteleiro M. (1991) A general methodologycal analysis for optimum design. International Journal for Numerical Methods in Engineering, Vol. 31, 85–111.
20. Navarrina F. et al. (200) High order shape design sensitivity: A unified approach. Computer Methods in Applied Mechanics and Engineering, Vol. 188, 681–696.
21. Navarrina F. et al. (2001) An efficient MP algorithm for structural shape optimization problems. Computer Aided Optimum Design of Structures VII, S. Hernández and C.A. Brebbia (Eds.), WIT Press, Southampton, 247–256.
22. Navarrina F. et al. (2002) Optimización Topológica de Estructuras: Una formulación de mínimo peso con restricciones en tensión (in Spanish). Métodos Numéricos en Ingeniería V (Book and CD-ROM, ISBN: 84-95999-03-X), SEMNI, Barcelona.
23. Navarrina F. et al. (2004) Topology optimization of structures: a minimum weight approach with stress constraints. Advances in Engineering Software, Vol. 36, 599–606.
24. París et al. (2005) A minimum weight FEM formulation for Structural Topological Optimization with local stress constraints. (WCSMO6) Proc. VI World Congress on Structural and Multidisciplinary Optimization. Rio de Janeiro, Brazil.
25. Pereira J. T. et al. (2004) Topology Optimization of Continuum structures with Material Failure Constraints. Structural and Multidisciplinary Optimization, Vol. 26 (1-2), 50–66.
26. Ramm E. et al. (2000) Advances in structural optimization including nonlinear mechanics. Proc. of the European Congress on Computational Methods in Applied Sciences and Engineering [ECCOMAS 2000] (CD-ROM, ISBN: 84-89925-70-4), ECCOMAS, Barcelona.
27. Rozvany G. I. N., Zhou M. (1991) Applications of the COC algorithm in layout optimization. In: Eschenauer H., Matteck C., Olhoff N. (Eds) Engineering Optimization in Design Processes, Proc. Int. Conf. held in Karlsruhe, Germany, Sept. 1990, pp. 59-70, Springer-Verlag, Berlin.
28. Rozvany G. I. N., Zhou M., Birker T. (1992) Generalized shape optimization without homogeneization. Structural Optimization, Vol. 4, 250–254.
29. Schmit L. A. (1960) Structural design by systematic synthesis. Proc. of the second ASCE Conference on Electronic Computation. Pittsburgh, PA: ASCE; 105-22.
30. Sigmund O. et Al. The TOPOPT homepage, www.topopt.dtu.tk
31. Stolpe M. and Svanberg K. (2001) On the trajectories of the epsilon-relaxation approach for stress-constrained truss topology optimization. Structural and Multidisciplinary Optimization, Vol. 21 (2), 140–151.
32. Yang R. J. and Chen C. J. (1996) Stress-based topology optimization. Structural Optimization, Vol. 12, 98–105.
33. Zhou M., Rozvany G. I. N. (1991) The COC algorithm, Part II: topological, geometrical and generalized shape optimization. Comp. Meth. Appl. Mech. Eng. Vol. 89, 309–336.