Improvements in the treatment of stress constraints in structural topology optimization problems☆

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Abstract

Topology optimization of continuum structures is a relatively new branch of the structural optimization field. Since the basic principles were first proposed by Bendsøe and Kikuchi in 1988, most of the work has been dedicated to the so-called maximum stiffness (or minimum compliance) formulations. However, since a few years different approaches have been proposed in terms of minimum weight with stress (and/or displacement) constraints.

These formulations give rise to more complex mathematical programming problems, since a large number of highly non-linear (local) constraints must be taken into account. In an attempt to reduce the computational requirements, in this paper, we propose different alternatives to consider stress constraints and some ideas about the numerical implementation of these algorithms. Finally, we present some application examples.

Key words:
Topology optimization of structures, stress constraints, constraints aggregation, minimum weight, parallel computing, perimeter penalization

1. Introduction

Topology optimization of structures is a relatively recent discipline in the field of structural optimization. Since the first model was introduced a lot of effort has been dedicated to deal with this problem. However, most of the works about this topic has been driven to maximum stiffness formulations due to computational reasons, among other considerations. More recently, different approaches with stress constraints have been proposed due to the important advantages that they offer (avoids checkerboard solutions, guarantees the feasibility of the solution, ...). However, the computational requirements are more restrictive in these formulations since the underlying optimization problem is much more complicated.

In this paper we present and compare three different approaches of the stress constraints for the topology optimization of structures problem: the local approach, the global approach and

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the block aggregated approach. We also introduce some important considerations in order to reduce their computational requirements, and we discuss some theoretical aspects like the mesh dependency or the singularity phenomena.

2. Topology optimization problem

The minimum weight with stress constraints topology optimization problem can be written, according to [1], as:

$$\begin{array}{lll} \textit{Find} & \boldsymbol{\rho} = \{\rho_e\}, & e = 1, \ldots, N_e \\ \textit{that minimizes} & F(\boldsymbol{\rho}) & \\ \textit{verifying} & g_j(\boldsymbol{\rho}) \leq 0, & j = 1, \ldots, m \\ & 0 < \rho_{min} \leq \rho_e \leq 1, & e = 1, \ldots, N_e \end{array} \tag{1}$$

where the design variable ρ_e is the relative density of element e (assumed uniform within the element), $F(\boldsymbol{\rho})$ is the objective function and g_j are the stress constraints of the problem. N_e is the total number of elements in the mesh and m is the number of constraints imposed. The value of ρ_{min} is slightly higher than zero to avoid numerical difficulties since the stiffness matrix would become singular. The model of microstructure used is equivalent to the SIMP model (Solid Isotropic Material with Penalty) but without any penalization (see [1]). The penalization of the intermediate densities is included in the objective function as

$$F(\boldsymbol{\rho}) = \sum_{e=1}^{N_e} (\rho_e)^{\frac{1}{p}} \int_{\Omega_e} \gamma_{mat} \, d\Omega, \tag{2}$$

where Ω_e is the domain of element e, γ_{mat} is the density of the material and $p \geq 1$ is the penalization parameter of the intermediate densities used to favor a mainly binary (0-1) distribution of material [1].

3. Stress constraints

In order to consider stress constraints we propose three different formulations: the local approach, the global approach and the block aggregated approach. The local approach imposes one stress constraint in the central point of each element of the mesh [1, 2, 3, 4, 5]. This local stress constraint can be defined as

$$g_e(\boldsymbol{\rho}) = \left[\widehat{\sigma}\left(\boldsymbol{\sigma}_e^h(\boldsymbol{\rho})\right) - \widehat{\sigma}_{max} \, \varphi_e\right](\rho_e)^q \leq 0 \quad \text{being} \quad \varphi_e = 1 - \varepsilon + \frac{\varepsilon}{\rho_e}, \tag{3}$$

where g_e is the stress constraint of element e and $\widehat{\sigma}$ is the reference stress used (usually the Von Mises criterion) obtained by means of the calculated stress tensor $\boldsymbol{\sigma}_e^h$ in the central point of the element. In order to avoid singularity phenomena when the relative density tends to zero, this constraint has been relaxed by using the function φ_e [2, 6]. The "relaxation parameter" ε usually takes values between 0.001 and 0.1. In addition, the exponent q allows to deal with constraints imposed on the homogenized stress tensor (when $q = 0$) or with constraints imposed on the effective stress tensor (when $q = 1$). According to [1] and [3], the use of effective stress reports important advantages since it reduces the non-linearity of the stress constraints when the relative density tends to zero.

The local approach of stress constraints usually requires to impose a high number of constraints due to the large number of elements (and design variables) involved. Consequently, this approach requires very large computing resources. Due to this fact, several alternative formulations have been developed in order to reduce the computing effort required.

We propose to use a global function that aggregates the effect of all the local constraints [3]. This global function was first proposed by Kreisselmeier-Steinhauser (and later used in [7], for example). Thus, the global constraint can be defined as

$$G_{KS}(\boldsymbol{\rho}) = \left[\frac{1}{\mu}\ln\left(\sum_{e=1}^{N_e} e^{\,\mu(\widehat{\sigma}_e^* - 1)}\right) - \frac{1}{\mu}\ln(N_e)\right] \leq 0 \tag{4}$$

being

$$\widehat{\sigma}_e^* = \frac{\widehat{\sigma}(\boldsymbol{\sigma}_e^h(\boldsymbol{\rho}))}{\widehat{\sigma}_{max}\ \varphi_e}, \tag{5}$$

where μ is the aggregation parameter and it usually takes values between 15 and 40 [3, 5]. Values of μ smaller than 15 allows an excessive violation of the local constraints and, on the other hand, values of μ higher than 40 produces a highly non-linear function. N_e is the number of stress constraints aggregated in the global function.

This approach reduces enormously the computing effort required but it also leads to a loss of information in the sensitivity analysis due to the constraints aggregation.

For this reason, we also propose a different strategy that establishes groups of elements that we call blocks (figure 1). Each block contains approximately an equal number of elements.

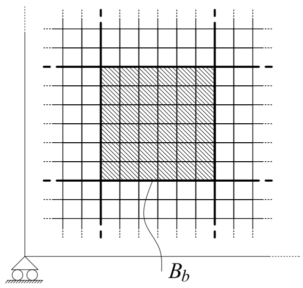

Figure 1: Example of block definition

The main idea of this approach is to impose one global constraint over the elements of each block. The global function used is the KS function proposed in (4). Thus,

$$G_{KS}^b(\boldsymbol{\rho}) = \left[\frac{1}{\mu}\ln\left(\sum_{e\in B_b} e^{\,\mu(\widehat{\sigma}_e^* - 1)}\right) - \frac{1}{\mu}\ln(N_e^b)\right] \leq 0, \tag{6}$$

where N_e^b is the number of elements aggregated in block b and B_b is the set of elements in block b.

This approach allows to define the number of blocks to use and consequently the number of stress constraints. Thus, this formulation is more general than the local one or the global one and includes them as a particular case [5, 8].

The number of blocks used and the shape of the blocks are the most important features of this formulation. However, we have observed in practice that the shape of the blocks does not have a considerable influence on the final solution. The number of blocks and the aggregation parameter are much more relevant.

In the application examples presented in this paper, the elements of each block present correlative numbers in the finite element mesh. This block definition algorithm usually produces deformed long blocks for the most usual finite element meshes used in topology optimization of structures. More compact procedures for the block definition (like the observed in figure 1) could lead to a more efficient problem. However, as it was mentioned before the shape of the blocks is not a critical issue.

4. Mesh dependency

The most usual formulations in topology optimization are subjected to mesh dependency phenomena. The origin of these phenomena is based on the fact that the original discrete statement is ill-posed. This problem is partially overcome by using a porous material with a predefined microstructure of material [1]. The most usual microstructure of material is the SIMP model (Solid Isotropic Material with Penalty). However, the use of the SIMP model does not guarantee the mesh independency. In maximum stiffness formulations, mesh dependency phenomena are directly associated to checkerboard layouts since the refinement of solutions with checkerboard distributions of material artificially increases the stiffness of the solutions. Thus, the refinement of the mesh increases the stiffness of the solution although the material distribution does not change substantially.

Minimum weight with stress constraints approaches (like the proposed in this paper) avoid checkerboard layouts due to the stress constraints and consequently mesh dependency phenomena are also removed. However, the refinement of the mesh usually produces more complicated distributions of material in local areas of the domain. These distributions slightly reduce the objective function but increases enormously the complexity of the solution. Consequently, these solutions are unwanted in practice and it is necessary to introduce some modifications to obtain solutions with a more reduced number of elements (trusses).

The most usual techniques developed to deal with the mesh dependency phenomena are, obviously, associated to maximum stiffness formulations. In order to avoid these mesh dependency phenomena, several procedures have been proposed: image filtering techniques [9], constraints over the gradient of the design variables [10], perimeter constraints [11], ... All these techniques perform well-posed formulations for the maximum stiffness topology optimization problem.

In this paper we introduce a penalization on the perimeter of the structure in order to reduce the complexity of the optimum solutions. Thus, the influence of the perimeter is included as a penalization in the objective function defined in section 2. The perimeter function presented is based on the total variation function (TV) proposed by Haber, *et al.* [11]:

$$TV(\rho) = \int_{\Omega\backslash\Gamma_J} \|\nabla\rho\| \, d\Omega + \int_{\Gamma_J} |< \rho >| \, d\Gamma_J \tag{7}$$

where $\Omega = \bigcup_\alpha \Omega_\alpha$ being Ω_α the set of disjointed regions (finite elements) that defines the whole structure domain Ω. The expression $|< \rho >|$ indicates the absolute difference of relative density between two neighbour disjunct regions Ω_α (finite elements).

If we impose that the relative density is constant for each element, the first term of equation (7) is null and the objective function of the topology optimization problem can be defined as:

$$F = \sum_{i=1}^{N_e} \int_{\Omega_i} (\rho_i)^{1/p} d\Omega_i + \eta \sum_{\Gamma_J} |< \rho >| L_J \tag{8}$$

where L_J is the length of the frontier between two contiguous elements and η is the weight factor between the cost of the structure and the perimeter. This weight factor can be determined by taking into account the values of the objective function and the perimeter. This factor is determined as a reduced percentage of the relation between the initial weight of the structure and the initial perimeter. This percentage usually varies from 1 % to 5 %. High values of this percentage avoids the generation of trusses in the optimal solution. Thus, great areas with intermediate densities appear. On the other hand, low values introduce an insignificant effect of the perimeter penalization.

5. Optimization algorithm

According to the approaches introduced in the previous section, the topology optimization of structures with stress constraints leads to mathematical programming problems type (1) with a large number of highly non-linear constraints type (3), (4) or (6) and a non linear objective function. An improved SLP algorithm with quadratic line-search seems to be a right choice to solve this kind of problems [1, 4, 12]. Thus, the linear approximation to problem (1) is stated (with additional side constraints) and solved at each iteration by means of the Simplex method [13]. This algorithm has demonstrated to work properly even if the global approach is used (only one constraint) [14]. The inactive constraints are disregarded, with the aim of saving computational resources. The required sensitivity analysis can be computed analytically. Full set of first order derivatives of the stress constraints are obtained via the adjoint variable method in order to reduce the computational effort. These derivatives are involved in the calculation of the search direction by means of the Simplex Algorithm. However, the second order directional derivatives of the stress constraints are computed analytically via a direct differentiation technique [5]. With this procedure, directional derivatives of all the stress constraints can be obtained although the full set of first order derivatives has not been calculated. Directional derivatives are required to develop a directional second order Taylor expansion used in the Quadratic Line Search.

6. Parallel computing

The computational effort required to solve the optimization problem proposed in [1] means an important limitation to this technique nowadays. However, some computational performances can be developed in order to reduce the computing time.

In section 5 some fundamental aspects about sensitivity analysis have been introduced in order to reduce the computational effort required. However, a better performance of this methodology can be obtained by computing the required derivatives in parallel. The number of constraints is usually very large for the local approach and the computation of the full set of first order derivatives of each stress constraint can be obtained separately. Thus, the computation of the first order stress sensitivities can be done in parallel by using all the available processors.

The parallelization of the full first order sensitivity analysis produces a very good speed-up for the local approach of stress constraints, reaching almost the maximum theoretical value.

However, to obtain an adequate performance of the whole process it is also necessary to parallelize the optimization algorithm.

The Simplex algorithm is an iterative procedure and, consequently, the parallelization of the whole process is not possible. On the other hand, the modification of the matrix of the problem at each inner iteration usually requires more than 95 % of the total computing time of the algorithm and can be easily parallelized. As it was expected, the speed-up of this algorithm is worse than the one obtained for the sensitivity analysis. Figure 2 shows the total speed-up obtained for the cantilever beam example [15] with 7200 elements and 7200 stress constraints by using the local approach. The parallel code was developed by using OpenMP directives in a Fortran source code. The calculations were carried out in a computing node with four dual core processors.

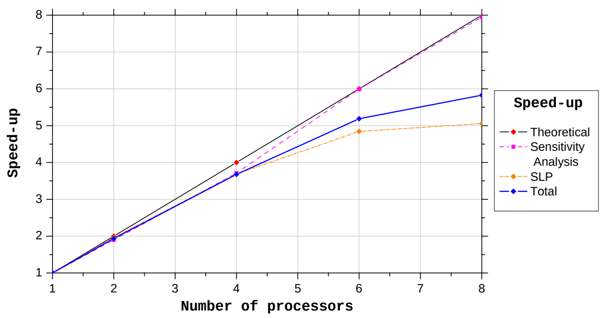

Figure 2: Total speed-up

7. Application examples

In this section, we present two structural problems frequently analyzed in topology optimization: the Michell beam with a centered load and the MBB beam. These examples are 2D structures in plane stress but we show three dimensional figures by assuming the relative density to be the thickness to better understand the solutions obtained.

7.1. Michell beam with a central force

The first example corresponds to the topology optimization of the Michell beam with a central load. Only the right half of the structure is analyzed due to the symmetry (see figure 3).

This example is a validation problem since the theoretical solution was proposed by Michell in 1904 [16]. Figure 4 shows the optimal material distribution for the topology optimization problem proposed.

The structure proposed is 1 cm thick and the total load applied is P=25 kN distributed into 4 elements around the central point of the domain. This example is solved with the local approach of the stress constraints using an initial mesh of 1800 eight-node quadrilateral elements. In this case it is not necessary to use neither the global constraint approach nor the block aggregation of the stress constraints since the size of the problem is not a limitation in computational terms.

Figure 5 shows the optimum solution obtained with the local approach of the stress constraints.

This optimal solution can be used to obtain a new refined mesh by removing the elements with relative density smaller than $\rho \leq 0.002$ and by dividing each one of the rest of the elements

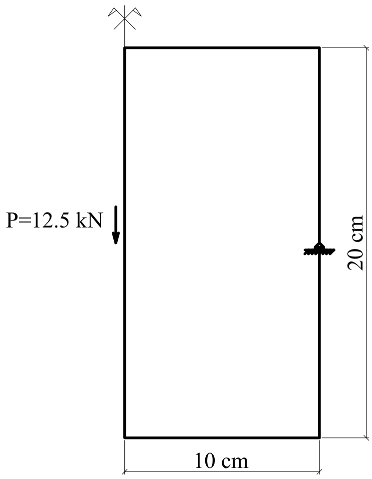

Figure 3: Geometry and applied loads of the Michell beam example

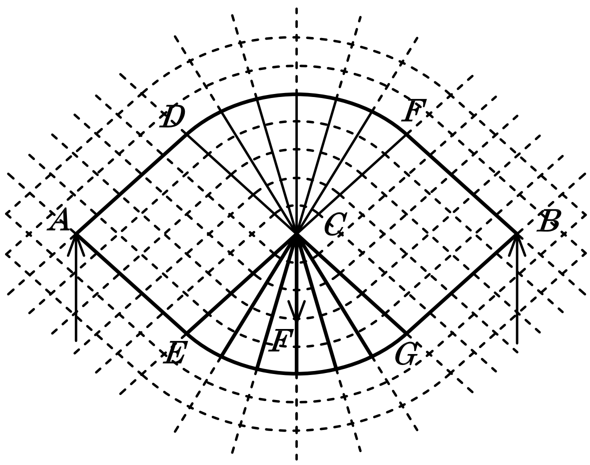

Figure 4: Analytic optimal solution proposed by Michell [16]

in four new ones (Figure 6 left). Figure 6 (right) shows the solution obtained with this refined mesh. This refinement technique allows to solve larger problems with a reduced number of elements and, consequently, with smaller computing resources.

Table 1 shows the most important parameters of the problem. The optimal volume of material and the optimal weight obtained with the formulation proposed are also presented and compared with the optimal ones obtained by Michell in [16]. In addition, the CPU time has been also analyzed in order to show the computational effort required.

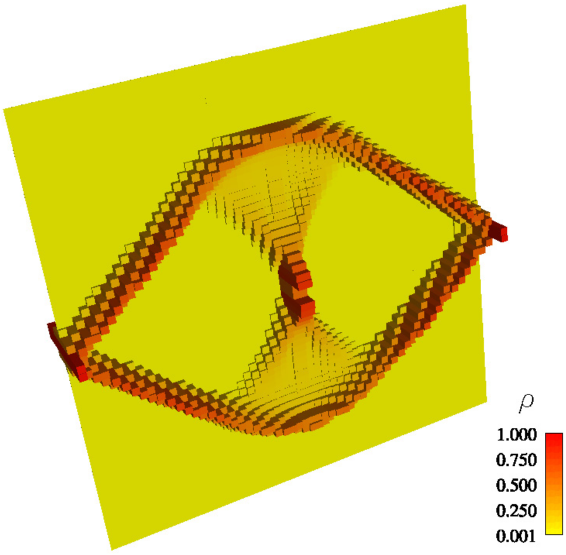

Figure 5: Michell beam solution with the local approach of the stress constraints, [ε = 0.004, p = 2, η = 0.005]

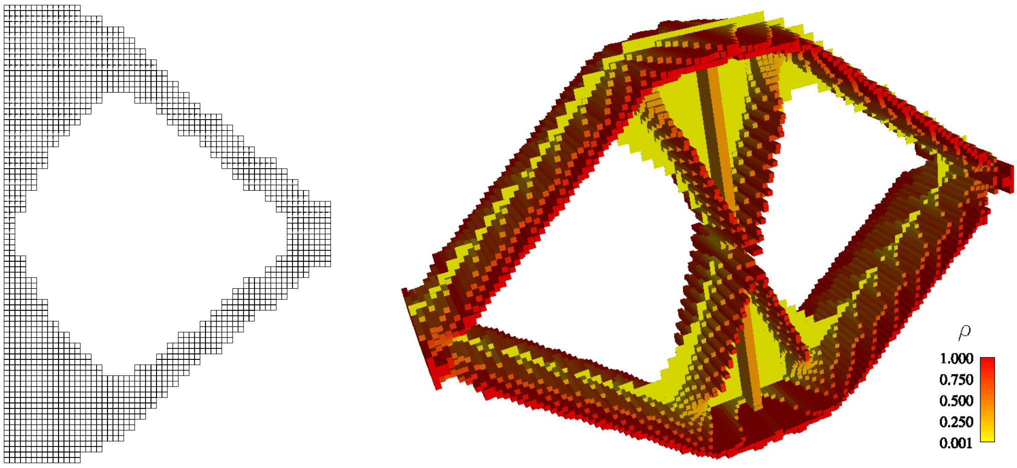

Figure 6: Refined mesh of the MBB beam obtained by removing all the elements of the solution proposed in figure 5 with $\rho_e \leq 0.002$ (left). Optimal solution obtained by using the local approach of the stress constraints and the refined mesh, [ε = 0.005, p = 2, N_e = 1688, η = 0.005] (right)

7.2. MBB beam

The second example corresponds to a classic MBB-type beam with sliding supports [15]. Only half of the structure is analyzed due to the symmetry. Figure 7 shows the dimensions of the domain and the position of the external load. Self-weight is considered. The domain of the structure is discretized in $N_e = 120 \times 40 = 4800$ eight-node quadrilateral elements. The

Michell Beam with a central load	Local Approach Fig. 5	Local App. Rem. Fig. 6 (right)
Number of elements	1800	1688
Number of iterations	406	404
Penalization (p)	2	2
Relaxation (ε)	0.004	0.005
Final weight (kN)	$2.02\ 10^{-3}$	$2.16\ 10^{-3}$
Final Volume (m^3)	$2.64\ 10^{-5}$	$2.82\ 10^{-5}$
Theoretical Volume (m^3)	$2.79\ 10^{-5}$	$2.79\ 10^{-5}$
Computing time ($s \times 10^3$)	47.2	91.9

Table 1: Summary of the most important parameters and results of the Michell beam with a central load problem

material being used is steel with density $\gamma_{mat} = 7650\ \mathrm{kg/m^3}$, Young's modulus $E = 2.1\ 10^5$ MPa, Poisson's ratio $\nu = 0.3$ and elastic limit $\widehat{\sigma}_{max} = 230$ MPa. The thickness of the structure is 1 m

This example is solved with the three formulations of stress constraints proposed in section 3 in order to compare the solutions obtained with them. Figures 7 (right), 8 (left) and 8 (right) show the solutions obtained with the local, the global and the block aggregated approaches of the stress constraints.

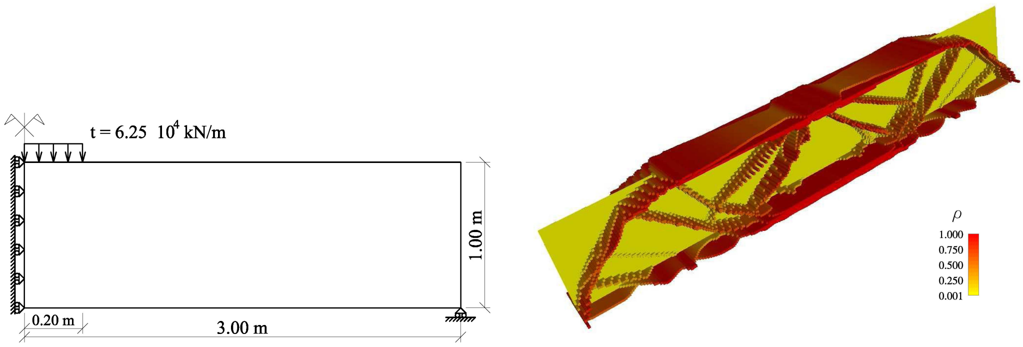

Figure 7: Geometry of the MBB beam example (left) and MBB solution with the local approach of the stress constraints, [$\varepsilon = 0.01$, $p = 4$, $\eta = 0$] (right)

Table 2 shows the most important parameters of the problem and the minimum weight obtained in the optimal solution. In addition, the CPU time has been also analyzed in order to show the computational effort required with all the formulations in order to analyze and compare them.

8. Conclusions

Structural Topology optimization with stress constraints is not a usual branch in the topology optimization field. However, these formulations offer important advantages versus maximum stiffness approaches since they avoid checkerboard layouts and present a more realistic objective

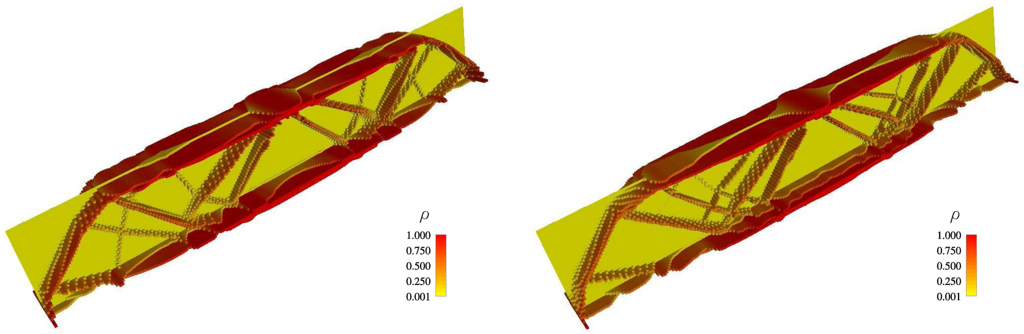

Figure 8: MBB solution with the global approach of the stress constraints, [ε = 0.01, p = 4, μ = 40, η = 0] (left) and with the block aggregated approach of the stress constraints, [ε = 0.02, p = 4, μ = 40, N_e^b = 60, η = 0] (right)

MBB beam	Local Appr. Fig. 7 right	Global Appr. Fig. 8 left	Block Aggr. Fig. 8 right
Number of elements	4800	4800	4800
Number of constraints	4800	1	80
Number of iterations	182	772	1005
Penalization (p)	4	4	4
Relaxation (ε)	0.01	0.01	0.02
Aggregation (μ)	-	40	40
Final/Initial weight	15.41 %	13.62 %	14.24 %
Computing time (s$\times 10^3$)	759.9	4.1	38.4

Table 2: Summary of the most important parameters and results of the MBB beam problem

function from an engineering point of view. In addition, the feasibility of the final solutions is guaranteed.

In this paper we propose three different formulations to impose stress constraints. The most usual and reliable procedure is the local approach of stress constraints since one stress constraint per element is imposed. However, this methodology introduces a large number of constraints in the optimization problem when fine FEM meshes are used.

Due to this fact, two additional procedures are analyzed in order to reduce the computational effort required: the global approach and the block aggregated approach. The global approach imposes only one global constraint that aggregates the effect of all the local constraints. On the other hand, the block aggregation of elements is a more general methodology that includes both previous formulations as a particular case.

Thus, if a large number of design variables is used, the block aggregation of elements is the most appropriate technique due to computational considerations. However, if the computing time is not too much restrictive the local approach is the most reliable formulation.

This paper also addresses some important considerations in order to reduce the computation effort required since parallelization techniques have been introduced. In addition, a perimeter penalization was introduced in the objective function in order to simplify the solutions obtained.

Finally, it is important to remark that minimum weight with stress constraints formulations produce fully satisfactory results versus maximum stiffness approaches. Consequently, maximum stiffness formulations should be replaced by minimum weight with stress constraints formulations since they offer very important advantages and the computational effort required is not a drastic limitation nowadays.

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