Block Aggregation of Stress Constraints in Topology Optimization of Structures

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Abstract

Structural topology optimization problems have been traditionally stated and solved by means of maximum stiffness formulations. On the other hand, some effort has been devoted to stating and solving this kind of problems by means of minimum weight formulations with stress (and/or displacement) constraints. It seems clear that the latter approach is closer to the engineering point of view, but it also leads to more complicated optimization problems, since a large number of highly non-linear (local) constraints must be taken into account to limit the maximum stress (and/or displacement) at the element level. In this paper, we explore the feasibility of defining a so-called global constraint, which basic aim is to limit the maximum stress (and/or displacement) simultaneously within all the structure by means of one single inequality. Should this global constraint perform adequately, the complexity of the underlying mathematical programming problem would be drastically reduced. However, a certain weakening of the feasibility conditions is expected to occur when a large number of local constraints are lumped into one single inequality. With the aim of mitigating this undesirable collateral effect, we group the elements into blocks. Then, the local constraints corresponding to all the elements within each block can be combined to produce a single aggregated constraint per block. Finally, we compare the performance of these three approaches (local, global and block aggregated constraints) by solving several topology optimization problems.

Key words: structural topology optimization, FEM, minimum weight, stress constraints, local constraints, global constraint, block aggregated constraints.
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1 Introduction

Structural topology optimization problems have been traditionally set out in terms of maximum stiffness (minimum compliance) formulations. In this approach, the goal is to distribute a given amount of material in a certain region, so that the stiffness of the resulting structure is maximized (the compliance, or energy of deformation, is minimized) for a given load case [1,2]. Even though this approach is quite convenient, it also entails some serious drawbacks, mainly: multiple load cases can not be considered; self-weight is normally ignored; the result varies with the amount of material to be distributed; and the final design could be unfeasible in practice, since no constraints are imposed on stresses (and/or displacements). Moreover, the maximum stiffness problem is essentially ill-posed. Thus, the solution oscillates as the discretization refinement is increased, what gives raise to mesh-dependent checkerboard layouts. This difficulty can be partially overcome by introducing porous materials [1]. But, on a regular basis, a spread porous material distribution is considered an unwanted result. Hence, additional penalization and stabilization techniques and image filters must be employed to avoid numerical instabilities and/or unrealistic —or simply useless— final solutions [1].

The authors, as other research groups, are working since a few years in the possibility of stating this kind of problems by means of a FEM-based minimum weight with stress (and/or displacement) constraints approach. Obviously, the physical meaning of this approach is closer to the engineering point of view, while any kind of constraint under multiple load cases could also be considered.

The basic and most intuitive procedure to preclude excessively high stresses (and/or excessively large displacements) within all the structure consists in limiting the maximum stress (and/or displacement) at a series of given points within each element [3,4]. This is commonly referred to as the "local (statement of) constraints approach". Thus, one can easily state quite complete and realistic optimization problems. The optimized solutions seem to be correct from the engineering point of view and their appearance could be considered closer to the engineering intuition than the results provided by the maximum stiffness approach. Furthermore, neither stabilization techniques nor image filters seem to be necessary to preclude unwanted final results [4]. However, this also leads to more complicated optimization problems with much higher computational requirements, since a large number of highly non-linear (local) constraints must be taken into account to limit the maximum stress (and/or displacement) at the element level.

In this paper, we explore the feasibility of defining a so-called global constraint, which basic aim is to limit the maximum stress (and/or displacement) simultaneously within all the structure by means of one single inequality. This is commonly referred to as the "global (statement of) constraints approach". Should this global constraint perform adequately, the complexity of the underlying mathematical programming problem would be drastically reduced.

Nevertheless, the performance of the global constraints approach falls significantly when a large number of local constraints are lumped into one single inequality. For this reason, we will finally introduce the so-called "block aggregated (statement of) constraints approach".

The global constraint formulation that we will use hereafter is based on the Kreisselmeier–Steinhauser function [5,6,7,8].

2 The Optimization Problem

In terms of a FEM-based minimum weight with stress (and/or displacement) constraints formulation, the topology optimization problem can be written as [4]

$$\begin{array}{lll} \textit{Find} & \boldsymbol{\rho} = \{\rho_e\}, & e = 1, \ldots, N_e \\ \textit{that minimizes} & F(\boldsymbol{\rho}) & \\ \textit{verifying} & g_j(\boldsymbol{\rho}) \leq 0, & j = 1, \ldots, m \\ & 0 < \rho_{min} \leq \rho_e \leq 1, & e = 1, \ldots, N_e \end{array} \tag{1}$$

where the design variable ρ_e is the relative density of element number e (what is assumed constant within the element) and N_e is the total number of elements in the mesh. Thus, if $d\Omega$ is the volume of a differential region within element number e, the volume occupied by the porous material within the differential region will be $\rho_e d\Omega$. The lower limit for the relative density (ρ_{min}) is introduced to preclude the entire hollowing out of the elements (since the concepts of displacement, strain and stress become meaningless and the stiffness matrix could even be singular in such a case).

The objective function is defined as

$$F(\boldsymbol{\rho}) = \sum_{e=1}^{N_e} (\rho_e)^{\frac{1}{p}} \int_{\Omega_e} \gamma_{mat} \, d\Omega, \tag{2}$$

where Ω_e is the element number e, γ_{mat} is the density of the material (assumed constant), and $p \geq 1$ is a tuning parameter that can be adjusted to favor a mainly compact distribution of material (since the intermediate values of the relative density are increasingly penalized as the value of p grows) [4].

It seems quite obvious that any kind of constraint could be taken into account in the above stated optimization problem. For the seek of simplicity, further discussion and examples are restricted to considering stress constraints type

$$\widehat{\sigma}_{min} \leq \widehat{\sigma}\left(\boldsymbol{\sigma}_j^h(\boldsymbol{\rho})\right) \qquad \text{and/or} \qquad \widehat{\sigma}\left(\boldsymbol{\sigma}_j^h(\boldsymbol{\rho})\right) \leq \widehat{\sigma}_{max}, \tag{3}$$

where $\boldsymbol{\sigma}_j^h(\boldsymbol{\rho})$ are the FEM-computed components of the stress tensor at each given point P_j for the actual values of the relative densities $\boldsymbol{\rho}$. The details on the FEM formulation for the structural analysis problem with relative density can be found in [4]. Finally, $\widehat{\sigma}(\boldsymbol{\sigma})$ is the reference stress expression that corresponds to the failure criteria being used (which values are limited).

In the 2D examples presented in this paper we consider materials with equal tensile and compressive strength limits. Thus, $\widehat{\sigma}(\boldsymbol{\sigma})$ is the Von Mises reference stress expression and $\widehat{\sigma}_{max}$ is the elastic stress limit of the material [4]. Then the constraints considered in (1) can be written as

$$g_j(\boldsymbol{\rho}) = \widehat{\sigma}\left(\boldsymbol{\sigma}_j^h(\boldsymbol{\rho})\right) - \widehat{\sigma}_{max} \leq 0. \tag{4}$$

3 Local Statement of Stress Constraints

Without losing generality, let's suppose that one stress constraint is imposed at one given point per element. Then, the optimization problem takes the form

$$\begin{array}{lll} \textit{Find} & \boldsymbol{\rho} = \{\rho_e\}, & e = 1, \ldots, N_e \\ \textit{that minimizes} & F(\boldsymbol{\rho}) & \\ \textit{verifying} & g_e(\boldsymbol{\rho}) \leq 0, & e = 1, \ldots, N_e \\ & 0 < \rho_{min} \leq \rho_e \leq 1, & e = 1, \ldots, N_e \end{array} \tag{5}$$

This is commonly referred to as the "local (statement of) constraints approach".

However, stress constraints type (4) can exhibit the so-called "singularity phenomena", that is due to the discontinuous nature of the stress when the relative density tends to zero [9]. Briefly, reaching the optimum could call for removing all the material within a certain element Ω_e. However, the corresponding restriction type (4) could be more severely violated as we get closer to the optimum (that is, for decreasing values of ρ_e slightly greater than 0), since the stress could rise as the material is being removed (until the element is completely hollowed out). Under these conditions, the gradient of the constraint would be negative in the vicinity of the optimum. Thus, any consistent non linear programming algorithm would try to increase the relative density instead of reducing it, what precludes convergence to the exact solution of the problem [4]. Singularity phenomena have also been observed in some theoretical truss optimization problems [10] and in other fields of structural optimization [3]. For this reason, statements type (4) are not fully satisfactory and they must be rewritten some way. Following the ideas of several authors [3,4,10] we propose the alternative statement for the local stress constraint

$$g_e(\boldsymbol{\rho}) = \left[\widehat{\sigma}\left(\boldsymbol{\sigma}_e^h(\boldsymbol{\rho})\right) - \widehat{\sigma}_{max}\,\varphi_e\right](\rho_e)^q \leq 0, \tag{6}$$

being

$$\varphi_e = 1 - \varepsilon + \frac{\varepsilon}{\rho_e}. \tag{7}$$

When $q = 0$, limits are imposed on the stress. When $q = 1$, limits are imposed on the so-called effective stress [4], what helps to remove some singularities. On the other hand, the value of the "relaxation parameter" $\varepsilon \in [0.001, 0.1]$ must be reduced as we approach the optimum during the optimization process.

The solutions to problems type (5) with constraints type (6) seem to be correct from the engineering point of view and their appearance could be considered closer to the engineering intuition than the results provided by the maximum stiffness approach [4]. Furthermore, neither stabilization techniques nor image filters seem to be necessary to preclude unwanted final results. However, these optimization problems are much more complicated and they have much higher computational requirements than the ones emerging from the maximum stiffness approach, since we have to deal now with a large number of highly non-linear constraints type (6).

4 Global Statement of Stress Constraints

We explore now the feasibility of limiting the stress simultaneously within all the structure by means of one single inequality. Should this be possible, the optimization problem would reduce to

$$\begin{array}{lll} \textit{Find} & \boldsymbol{\rho} = \{\rho_e\}, & e = 1, \ldots, N_e \\ \textit{that minimizes} & F(\boldsymbol{\rho}) & \\ \textit{verifying} & G(\boldsymbol{\rho}) \leq 0, & \\ & 0 < \rho_{min} \leq \rho_e \leq 1, & e = 1, \ldots, N_e \end{array} \tag{8}$$

This is commonly referred to as the "global (statement of) constraints approach". Obviously, if the so-called global constraint $G(\boldsymbol{\rho})$ performs adequately, the complexity of the mathematical programming problem and the associated computational requirements (both the data storage and the computing time) would be drastically reduced in comparison with (5).

Therefore, the essential point is to define an adequate procedure for aggregating all the local constraints in a single global one. The global constraint formulation that we present hereafter is based on the Kreisselmeier–Steinhauser function [6,7,8], that is mainly being used at present in aero-structural optimization. Furthermore, we have introduced some modifications that improve the numerical performance of the resulting global constraint. The proposed global constraint takes the form

$$G_{KS}(\boldsymbol{\rho}) = \left[\frac{1}{\mu} \ln \left(\sum_{e=1}^{N_e} e^{\, \mu(\widehat{\sigma}_e^* - 1)} \right) - \frac{1}{\mu} \ln(N_e) \right] \leq 0 \tag{9}$$

being

$$\widehat{\sigma}_e^* = \frac{\widehat{\sigma}\Big(\boldsymbol{\sigma}_e^h(\boldsymbol{\rho}) \Big)}{\widehat{\sigma}_{max} \, \varphi_e}. \tag{10}$$

The use of the normalized reference stress $\widehat{\sigma}_e^*$ is intended to rescale the arguments of the exponential terms. In addition, it helps to prevent a possible overflow condition to occur.

On the other hand, μ is a tuning parameter that penalizes the failure to satisfy the local constraints. In theory, global constraint (9) becomes equivalent to the constraint with highest value in each iteration as μ tends to infinity. However, when the value of μ is too large, global constraint (9) can become too difficult to manage, both for practical and theoretical reasons. Thus, for increasing values of μ the non-linearity of the global constraint function is boosted; and overflow conditions are more likely to occur. Consequently, it becomes more difficult to obtain a reasonably good numerical solution to problem (8). On the other hand, global constraint (9) will not adequately represent the corresponding whole set of local constraints if the value of μ is not large enough. In such a case, the solution to problem (8) will not be satisfactory. Therefore, it is extremely important to assign a correct value to parameter μ.

Figures 1 and 2 depict the value of the global constraint for different values of parameter μ in different conditions. In Figure 1, $\widehat{\sigma}_e^* = 0.90$ at 50% of the elements. The curves compare the values of the global constraint for different values of $\widehat{\sigma}_e^*$ (assumed all equal) at the remaining elements. In Figure 2, $\widehat{\sigma}_e^* = 0.90$ at the elements in which the corresponding local constraint is satisfied, and $\widehat{\sigma}_e^* = 1.10$ at the elements in which the corresponding local constraint is violated. The curves compare the values of the global constraint for a growing percentage of violated local constraints. The plotted results can help to make a decision on how to adjust the correct value of parameter μ. When μ is not large enough, the global constraint could be satisfied although most of the local constraints were violated. On the other hand, it seems that it would not be necessary to take large values of μ. On a regular basis, it seems reasonable to adjust the value of μ between 20 and 30, or between 15 and 40 as much.

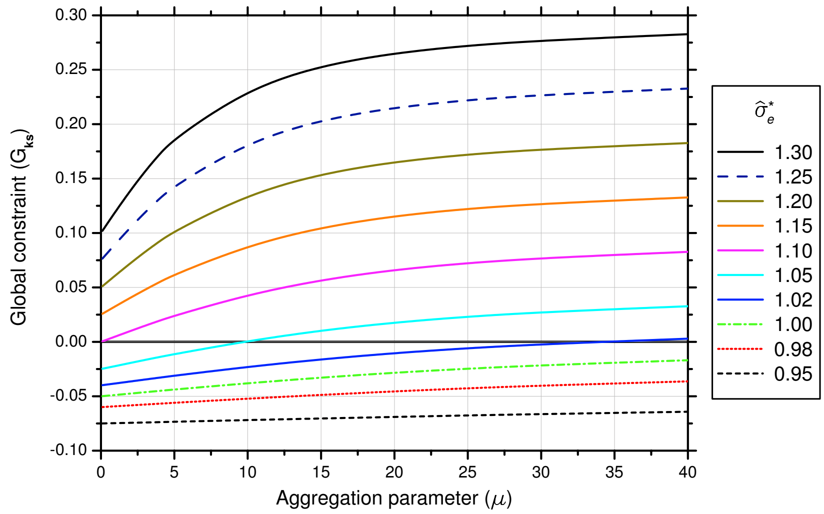

Figure 1. Global constraint versus μ for a growing value of $\widehat{\sigma}_e^*$.

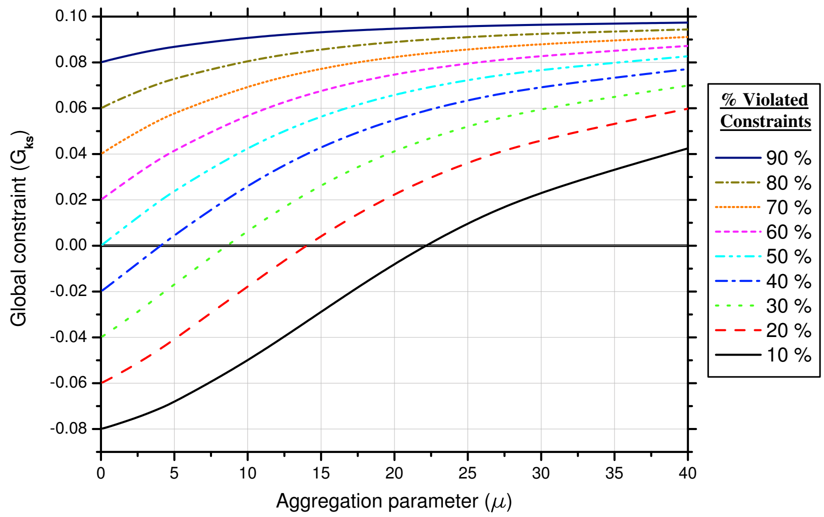

Figure 2. Global constraint versus μ for a growing % of violated constraints.

A certain loss of strictness is expected to occur in the fulfillment of the feasibility conditions when a large number of local constraints are lumped into one single inequality by means of the global constraints approach. For this reason, we will introduce a more suitable class of global type constraints by grouping the elements into blocks.

5 Block Aggregated Statement of Stress Constraints

Let us group the N_e elements into a relatively small number N_b of blocks or groups of elements $\{B_b\}_{b=1,N_b}$. As a general rule, we should group the elements taking into account its proximity in such a way that all the resulting blocks will contain a similar number of elements (see Figure 3). Then, the local constraints

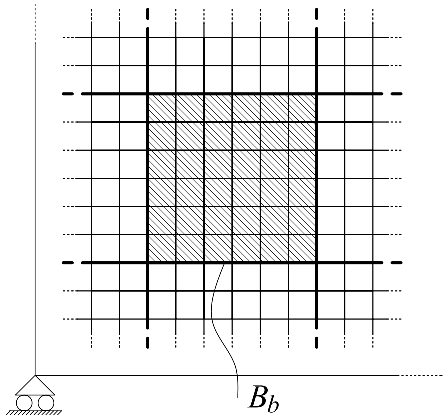

Figure 3. Block definition within the Finite Element mesh.

corresponding to all the elements within each block could be combined to produce a single inequality per block. Should this be done, the optimization problem would reduce to

$$\begin{array}{lll} \textit{Find} & \boldsymbol{\rho} = \{\rho_e\}, & e = 1, \ldots, N_e \\ \textit{that minimizes} & F(\boldsymbol{\rho}) & \\ \textit{verifying} & G^b(\boldsymbol{\rho}) \leq 0, & b = 1, \ldots, N_b \\ & 0 < \rho_{min} \leq \rho_e \leq 1, & e = 1, \ldots, N_e \end{array} \tag{11}$$

We will refer to the latter as the "block aggregated (statement of) constraints approach". Obviously, if the so-called block aggregated constraints $G^b(\boldsymbol{\rho})$ perform adequately and the number of blocks N_b is much smaller than the number of elements N_e, the complexity of the mathematical programming problem and the associated computational requirements (both the data storage and the computing time) would be drastically reduced in comparison with (5).

Again, the essential point is to define an adequate procedure for aggregating all the local constraints within each block into a single inequality. Thus, the block aggregated constraint formulation that we propose is just an adaptation of (9) and takes the form

$$G^b_{KS}(\boldsymbol{\rho}) = \left[\frac{1}{\mu} \ln \left(\sum_{e \in B_b} e^{\,\mu(\widehat{\sigma}^*_e - 1)} \right) - \frac{1}{\mu} \ln(N^b_e) \right] \leq 0 \tag{12}$$

where N^b_e is the number of elements contained in block b. As it was mentioned before, the use of the normalized reference stress $\widehat{\sigma}_e^*$ is intended to rescale the arguments of the exponential terms. In addition, it helps to prevent a possible overflow condition to occur. Again, it is extremely important to assign a correct value to parameter μ, and the same considerations that were previously exposed apply to this case.

It seems quite obvious what this strategy seeks for: we expect to retain the advantages of the global constraints approach, and at the same time we expect to mitigate its undesirable collateral effects by limiting the number of constraints being aggregated. Therefore, a large number of elements should not be grouped within each block.

For $N_b = 1$ problem (11) reduces to the global constraints approach (8), while for $N_b = N_e$, problem (11) reduces to the local constraints approach (5) taking $q = 0$ in expression (6). Therefore, as the number of blocks is raised we expect the results to be more precise, at the expense of growing computational requirements.

For a moderate, well adjusted number of blocks we expect the quality of the results to be as good as the ones given by the local constraints approach, with a significantly lower computational cost.

5.1 Block Definition

The correct grouping of the elements could be a key issue, and it could determine both the quality of the final results and the corresponding computational cost, and even the viability of the optimization process itself.

In the examples presented hereafter in this work the blocks have been created by just grouping the elements with correlative indexes in the Finite Element mesh. But there is not a reason why this strategy should be preferred to any other. In fact, this choice could produce most likely non-regular and deformed and/or non-connected patterns, even in the case of structured rectangular meshes. It is obvious that different techniques could be proposed to create more compact, well-shaped blocks, by grouping the elements in accordance with its proximity. On the other hand, a simple random draw would probably lead to a less biased distribution of the elements within the blocks.

In the numerical examples presented in this paper we have observed that the distribution of elements into blocks has only a slight influence over the final results. In our experience, the number of blocks being defined (and, thus, the number of elements aggregated into each block) and the value of parameter μ in expression (9) play a much more important role than the strategy adopted for aggregating the elements into a certain number of blocks for a given mesh.

It remains to be known whether this is a critical point in general or not, and further research should be devoted to this line in a close future.

6 Optimization Algorithms

In practice, the local constraints approach leads to mathematical programming problems type (5) with a large number of highly non-linear constraints type (6). An improved SLP algorithm with quadratic line-search seems to be the right choice to solve this kind of problems [4,11,12]. Thus, the linear approximation to problem (5) is stated (with additional side constraints) and solved at each iteration by means of the Simplex method [13]. The inactive constraints are disregarded, with the aim of saving computational resources. Even though the obtained results are quite promising [4,12], both the data storage and the computing time associated to stating and solving the underlying linear programming problems grow very fast with the number of elements N_e. This fact severely restricts the applicability of the technique.

On the other hand, the global constraints approach leads to mathematical programming problems type (8) with only one highly non-linear constraint type (9). To solve this kind of problems we propose the modified inverse barrier function

$$\phi(\boldsymbol{\rho}, r) = F(\boldsymbol{\rho}) \left[1 - r \frac{1}{G_{KS}(\boldsymbol{\rho})}\right]. \tag{13}$$

In comparison with the standard definition [14], the inverse of the global constraint in the above expression is multiplied times the objective function. We recall that the expression of $G_{KS}(\boldsymbol{\rho})$ type (9) is non-dimensional, unlike the expression of $F(\boldsymbol{\rho})$ type (2). The rescaling introduced by this product improves the numerical conditioning of the problem, while possible dimension conflicts are prevented. Furthermore, it helps to adequately calibrate the value of the so-called barrier parameter r.

In addition, it is convenient to include the side constraints of the design variables into the barrier function too, in order to avoid false convergence situations due to the possible truncation of the search direction. With this aim, we define the modified barrier function (including side constraints) as

$$\phi(\boldsymbol{\rho}, r) = F(\boldsymbol{\rho}) \left[1 - r \frac{1}{G_{KS}(\boldsymbol{\rho})} - r_l \sum_{e=1}^{N_e} \left(\frac{1}{\rho_{min} - \rho_e} + \frac{1}{\rho_e - 1} \right) \right], \tag{14}$$

where r_l is the barrier parameter corresponding to the side constraints.

Then, the quasi-unconstrained non-linear programming problem

$$\begin{array}{lll} \textit{Find} & \boldsymbol{\rho} = \{\rho_e\}, & e = 1, \ldots, N_e \\ \textit{that minimizes} & \phi(\boldsymbol{\rho}, r), & \\ \textit{verifying} & 0 < \rho_{min} \leq \rho_e \leq 1, & e = 1, \ldots, N_e \end{array} \tag{15}$$

is solved by means of the Fletcher-Reeves conjugate gradient method [14], which performance is improved by using a complementary quadratic line-search (where we take again the side constraints into account). On a regular basis, both the data storage and the computing time associated to stating and solving the underlying quasi-unconstrained non-linear programming problems grow linearly with the number of elements N_e. This fact expands the applicability of the technique far beyond the possibilities of the local approach.

Finally, the block aggregated constraints approach leads to mathematical programming problems type (11) with a moderate number of highly non-linear constraints type (12). Therefore, the improved SLP algorithm with quadratic line-search that was previously mentioned seems to be —once more— the right choice to solve this kind of problems [11].

7 Sensitivity Analysis

The three above presented approaches require to perform a full first order sensitivity analysis at each iteration. This is a critical issue that could compromise the viability of the whole optimization process, since the computing effort devoted to the sensitivity analysis could become absolutely unaffordable. In this work, the first order derivatives are computed by means of an analytical implementation of the adjoint state method [15,16,17]. Thus, we avoid storing a large amount of intermediate results while the computing effort devoted to solving linear systems is minimized.

On the other hand, the three approaches require an additional second order directional sensitivity analysis at each iteration. This is always done by means of an analytical implementation of the direct differentiation method [15,16,17], since data storing and computing time are not critical issues at this point.

Overall, we could say that the computational cost for the sensitivity analysis is indeed expected to be much lower (or even negligible) in the case of the global constraints approach than in the case of the local constraints approach. For a moderate number of blocks, the computational cost of the sensitivity analysis in the case of the block aggregated constraints could be up to a few times higher than in the case of the global constraints approach, but still much lower than in the case of the local constraints approach. The same considerations apply to the amount of data storage that is required, although the difference between the global approach and the block aggregated approach is less significant in this case.

8 Numerical Examples

The examples presented below have been solved by means of the three proposed approaches (local constraints, global constraints and block aggregated constraints) in order to compare the obtained solutions and their corresponding computing effort.

Actually, all the examples are two-dimensional. However, the solutions are represented as 3D solids in order to facilitate the understanding of the results, being the false thickness proportional to the relative density at each point.

8.1 L-shape Beam

The first example corresponds to an L-shape beam. The upper edge is built into the roof. Figure 4 shows the dimensions of the domain and the position of the external load. Self-weight is also considered. The L-shaped domain (1 m thick) containing the structure is discretized in $N_e = 6400$ eight-node quadrilateral elements. The material being used is steel with density $\gamma_{mat} = 7650\ \mathrm{kg/m^3}$, Young's modulus $E = 2.1\ 10^5$ MPa, Poisson's ratio $\nu = 0.3$ and elastic limit $\widehat{\sigma}_{max} = 230$ MPa.

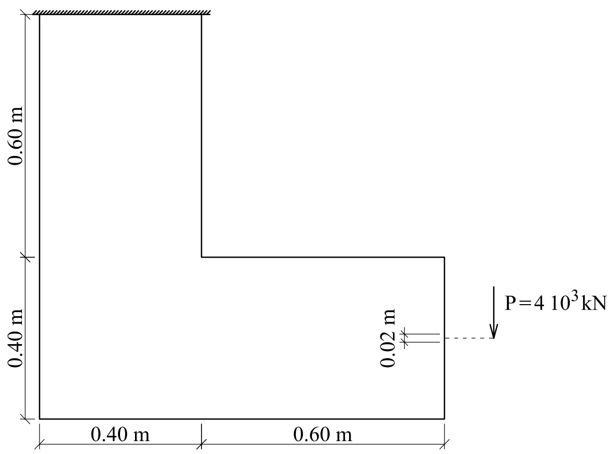

Figure 4. Example 1: L-shape beam. Domain definition and external loads.

Figure 5 shows the solution obtained by means of the local constraints approach for $p = 4$, $q = 1$ and $\varepsilon = 0.02$. Figure 6 shows the solution obtained by means of the global constraints approach (right) for $p = 4$, $q = 1$, $\varepsilon = 0.02$ and $\mu = 20$. Figure 7 shows the solution obtained by means of the block aggregated constraints approach for $p = 4$, $q = 1$, $\varepsilon = 0.02$, $\mu = 20$, $N_b = 100$ and $N_e^b = 64$. As it can be observed, the obtained solutions (see Figures 5, 6 and 7) are in accordance to the engineering experience and compare fairly well with the results previously obtained by Duysinx [3] and with the analytical solutions proposed by Lewinsky [18]. The analytical solution does not consider self-weight but, in this case, this fact does not substantially modify the final material distribution.

Table 1 compares the computational cost associated to solve this problem by means of the three above mentioned approaches.

Due to the large number of elements being used, the SLP algorithm described in section 6 has been used to solve all the underlying optimization problems in this case.

Table 1
Example 1: L-SHAPE beam. Computational cost.

L-SHAPE BEAM	Local Approach Fig. 5	Global Approach Fig. 6	Block Agg. Approach Fig. 7
Number of variables	6400	6400	6400
Number of constraints	6400	1	100
Number of iterations	253	530	341
Computing time (h)	271.4	7.1	24.4

8.2 MBB beam

The second example corresponds to a classic MBB-type beam with sliding supports [1]. Only half of the structure is analyzed, because of symmetry. Figure 8 shows the dimensions of the domain and the position of the external load. Self-weight is also considered. The rectangular domain (1 m thick) containing the structure is discretized in $N_e = 60 \times 20 = 1200$ eight-node quadrilateral elements. The material being used is steel with density $\gamma_{mat} = 7650$ kg/m^3, Young's modulus $E = 2.1\ 10^5$ MPa, Poisson's ratio $\nu = 0.3$ and elastic limit $\widehat{\sigma}_{max} = 230$ MPa.

Figure 9 shows the solutions obtained by means of the local constraints approach for $p = 4$, $q = 1$ and $\varepsilon = 0.02$. Figure 10 shows the solution obtained by means of the global constraints approach (right) for $p = 4$, $\varepsilon = 0.02$ and

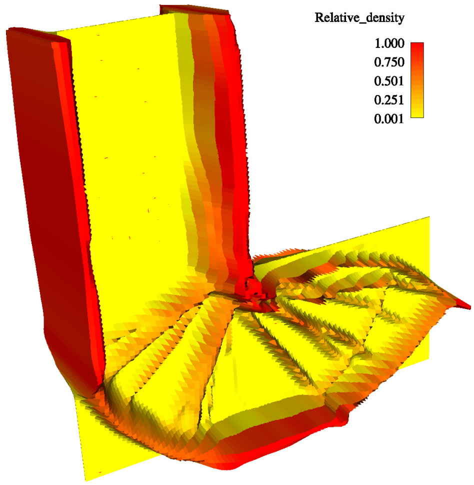

Figure 5. Example 1: L-shape beam. Distribution of material at the final solution. [Local constraints approach]

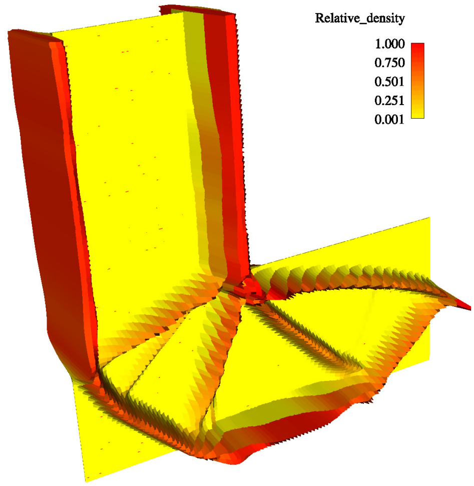

Figure 6. Example 1: L-shape beam. Distribution of material at the final solution. [Global constraints approach]

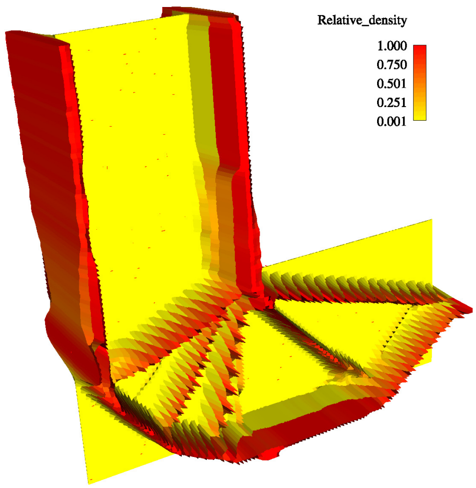

Figure 7. Example 1: L-shape beam. Distribution of material at the final solution. [Block aggregated constraints approach]

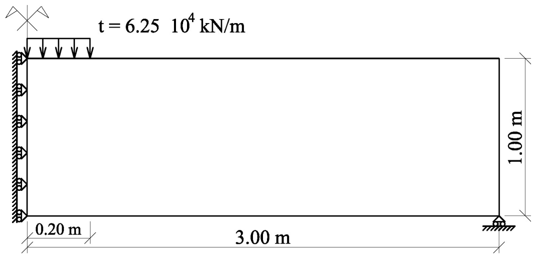

Figure 8. Example 2: MBB beam. Domain definition and external loads.

$\mu = 20$. Figure 11 shows the solution obtained by means of the block aggregated constraints approach for $p = 4$, $\varepsilon = 0.02$, $\mu = 20$, $N_b = 24$ and $N_e^b = 50$. As it can be observed, the obtained solutions (see figures 9, 10 and 11) are very similar to the analytic solution proposed by Lewinsky, *et al.*[19]. The analytical solution does not consider self-weight but, in this example, this fact does not substantially modify the final material distribution.

Table 2 compares the computational cost associated to solve this problem by means of the three above mentioned approaches.

Due to the high number of elements being used, the SLP algorithm described in section 6 has been used to solve the underlying optimization problems corresponding to the local approach (see figure 9) and to the block aggregated approach (see figure 11). The optimization problem corresponding to the global approach (see figure 10) has been solved by means of the barrier function algorithm described within the same section.

Table 2
Example 2: MBB beam. Computational cost.

MBB BEAM	Local Approach Fig. 9	Global Approach Fig. 10	Block Aggr. Approach Fig. 11
Number of variables	1200	1200	1200
Number of constraints	1200	1	24
Number of iterations	253	1111	404
Computing time (h)	3.91	0.18	0.14

8.3 Cantilever Support

The third example corresponds to a cantilever support. The structure is built into two circular holes located on the right side. Figure 12 shows the dimensions of the domain and the position of the external load. Self-weight is

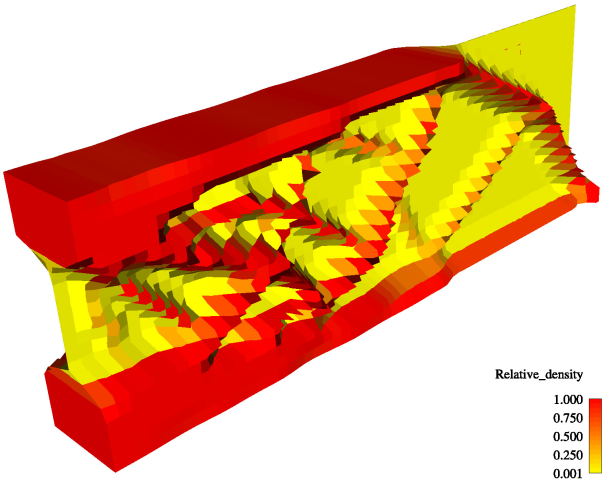

Figure 9. Example 2: MBB beam. Distribution of material at the final solution. [Local constraints approach]

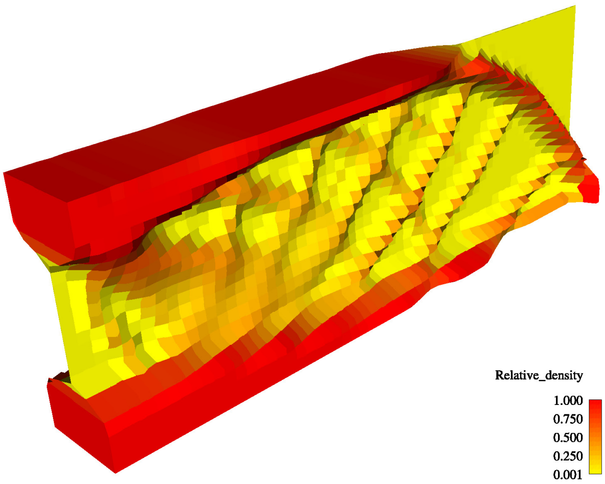

Figure 10. Example 2: MBB beam. Distribution of material at the final solution. [Global constraints approach]

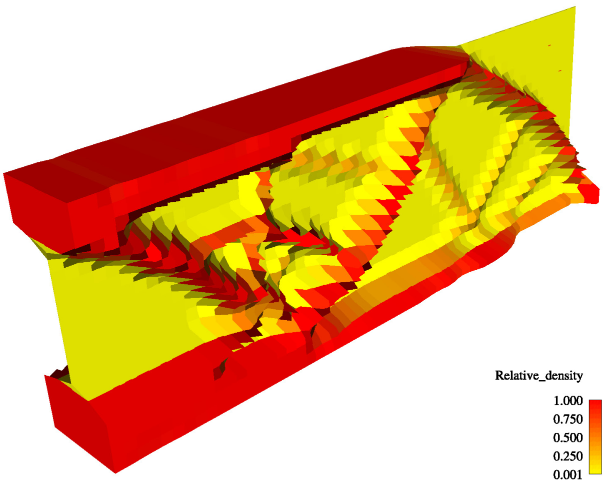

Figure 11. Example 2: MBB beam. Distribution of material at the final solution. [Block aggregated constraints approach]

also considered. The rectangular domain (1 cm thick) containing the structure is discretized in $N_e = 5808$ eight-node quadrilateral elements. The material being used is steel with density $\gamma_{mat} = 7650$ kg/m^3, Young's modulus $E = 2.1\ 10^5$ MPa, Poisson's ratio $\nu = 0.3$ and elastic limit $\widehat{\sigma}_{max} = 230$ MPa.

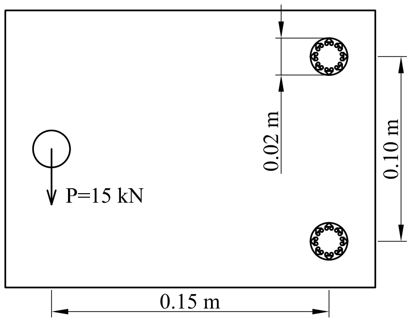

Figure 12. Example 3: Cantilever support. Domain definition and external loads.

Figure 13 shows the solution obtained by means of the local constraints approach for $p = 4$, $q = 1$ and $\varepsilon = 0.01$. Figure 14 shows the solution obtained by means of the global constraints approach for $p = 4$, $\varepsilon = 0.01$ and $\mu = 20$. Figure 15 shows the solution obtained by means of the block aggregated constraints approach for $p = 4$, $\varepsilon = 0.01$, $\mu = 20$, $N_b = 24$ and $N_e^b = 50$. This example was first analyzed by means of shape optimization techniques by Zhang [20] and more recently by means of topology optimization techniques (maximum stiffness approach) by Duysinx [21].

Table 3 compares the computational cost associated to solve this problem by means of the three above mentioned approaches.

Due to the high number of elements being used, the SLP algorithm described in section 6 has been used to solve all the underlying optimization problems in this case.

Table 3
Example 3: Cantilever support. Computational cost.

CANTILEVER SUPPORT	Local Approach Fig. 13	Global Approach Fig. 14	Block Aggr. Approach Fig. 15
Number of variables	5808	5808	5808
Number of constraints	5808	1	88
Number of iterations	253	430	507
Computing time (h)	157.0	19.7	27.0

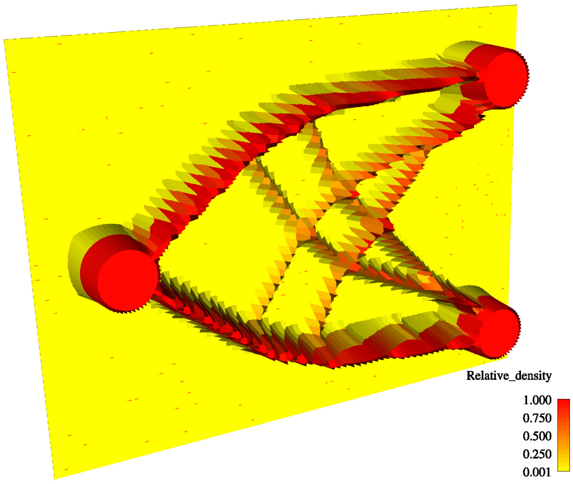

Figure 13. Example 3: Cantilever support. Optimal distribution of material. [Local constraints approach]

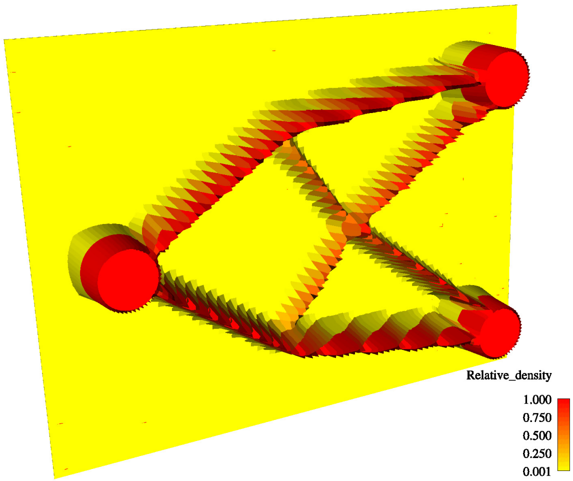

Figure 14. Example 3: Cantilever support. Optimal distribution of material. [Global constraints approach]

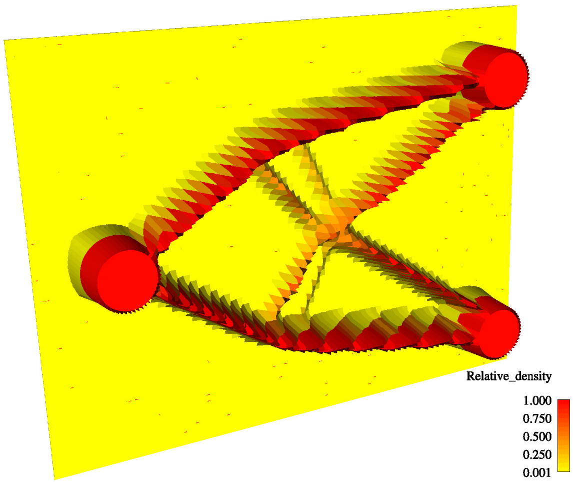

Figure 15. Example 3: Cantilever support. Optimal distribution of material. [Block aggregated constraints approach]

9 Conclusions

There are both conceptual and practical reasons that justify the interest in setting out structural topology optimization problems in terms of minimum weight formulations with a large number of highly non-linear local constraints. This is commonly referred to as the local constraints approach. However, the applicability of the technique is severely restricted by the extraordinarily high computational requirements of this kind of problems.

With the aim of reducing the complexity of these problems, we explore the feasibility of defining a so-called global constraint, which basic aim is to enforce the fulfillment of all the local constraints by means of one single inequality. This is commonly referred to as the "global (statement of) constraints approach". However, a certain loss of strictness in the fulfillment of the feasibility conditions is expected to occur when a large number of local constraints are lumped into one single inequality.

For this reason, we have introduced a more suitable class of global type constraints by grouping the elements into blocks. Then, the local constraints corresponding to all the elements within each block could be combined to produce a single aggregated constraint per block. We refer to the latter as the "block aggregated (statement of) constraints approach". In this way we expect to retain the advantages of the global constraints approach while its undesirable collateral effects could be partially mitigated.

In this paper, we have proposed specific procedures for correctly stating local constraints (local constraints approach) and for constraint aggregation (global constraints and block aggregated constraints approaches) in structural topology optimization problems. The performance of these three approaches has been tested and compared by stating and solving some application examples.

The computational requirements (both the data storage and the computing time) have been an order of magnitude lower for the global constraints approach and for the block aggregated constraints approach when compared to the local constraints approach, as it was expected. In return, the results have not been exactly equivalent, but quite similar. On the other hand, the block aggregated constraints approach gets to retain the advantages of the global constraints approach while its undesirable collateral effects are greatly mitigated, in return for a small increase in the computational cost of the technique.

The important reduction in the computational cost due to the constraint aggregation clearly compensates for the slight loss of accuracy in the results. Moreover, the applicability of the technique can be expanded this way far beyond its original possibilities.

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