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# GROUP TOPOLOGIES ON VECTOR SPACES AND CHARACTER LIFTING PROPERTIES

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ABSTRACT. It is known that every continuous character on a topological vector space can be lifted to a continuous linear functional and, moreover, these liftings give rise to a topological isomorphism between the dual group and the dual space, when both are endowed with the compact-open topology. We investigate the presence of these properties in more general topologized real vector spaces.

## 1. Preliminaries

In Functional Analysis, the study of not necessarily linear, additive group topologies on vector spaces can be traced back at least to the fifties. For instance, topological vector groups (as defined below) were introduced by Raĭkov ([19]), but the notion had been implicitly used in earlier references.

The first insights on the group duality of topological vector spaces are from about the same time (the words "group duality" meaning the use of the one-dimensional torus, rather than the real line, as the dualizing object). M. F. Smith proved in [20] that every continuous character on a topological vector space can be lifted to a continuous linear functional and, moreover, these liftings result in a topological isomorphism between the dual group and the dual space, both endowed with the compact-open topology. (The first part of this assertion was proved independently by Hewitt and Zuckermann in [14].)

These definitions and results were put together and given a new prominence in W. Banaszczyk's theory of nuclear groups ([2]). His work paved the way to proving significant generalizations of several basic theorems from abstract harmonic analysis and duality theory of topological Abelian groups, which were carried out by himself and other authors. The theory makes essential use of locally convex vector groups and other classes of topologized vector spaces.

In this paper we first give a survey of different topological properties that may be present in a topological Abelian group which algebraically is a real vector space. Next we explore the way these properties are related to the possibility of lifting continuous characters to continuous linear functionals on these spaces. Note that in the topological group framework, Nickolas ([18]) showed that the related problem of lifting continuous characters to continuous real valued group homomorphims is relevant in the study of the structure of dual groups.

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For any Abelian group $G$, any set $U \subset G$ and any $n \in \mathbb{N}$ we define

$$U_{(n)} = \bigcap_{k=1}^{n} \{x \in G : kx \in U\}$$

Note that if $G$ is a topological Abelian group and $U$ is a neighborhood of zero in $G$, the sets $U_{(n)},\ n \in \mathbb{N}$ are neighborhoods of zero as well.

The symbol $\mathbb{T}$ will denote the multiplicative group of all complex numbers with modulus 1, endowed with the topology induced by the usual one on $\mathbb{C}$. The canonical covering projection $[t \in \mathbb{R} \mapsto \exp(2\pi i t) \in \mathbb{T}]$ will be denoted by $p$. We will also use a special notation for the following distinguished neighborhood of 1 in $\mathbb{T}$:

$$\mathbb{T}_+ = p([-1/4, 1/4]) = \{t \in \mathbb{T} : \operatorname{Re} t \geq 0\}$$

The following fact is standard (for a proof see e. g. [2], 1.2):

Proposition (1.1). *For every $n \in \mathbb{N}$,* $\quad (\mathbb{T}_+)_{(n)} = p([-\frac{1}{4n}, \frac{1}{4n}])$.

For a given topological Abelian group $G$, let us denote by $\mathcal{N}_0(G)$ the family of all neighborhoods of the unit element $0 \in G$, in the case of additive notation. From Proposition (1.1) it is not difficult to derive the following known result:

Proposition (1.2). *Let $G$ be a topological Abelian group and $\chi \colon G \to \mathbb{T}$ a group homomorphism. Then $\chi$ is continuous if and only if there exists $U \in \mathcal{N}_0(G)$ such that $\chi(U) \subset \mathbb{T}_+$.*

Group homomorphisms from $G$ to $\mathbb{T}$ are usually called *characters* of $G$. We shall denote by $G^\wedge$ the set of all continuous characters of the topological Abelian group $G$. Note that $G^\wedge$ is an Abelian group under pointwise product of characters.

Let $G$ be a topological Abelian group, and $U$ a subset of $G$. The set of all continuous characters $\chi \in G^\wedge$ such that $\chi(U) \subset \mathbb{T}_+$ is called the *polar* of $U$ and denoted by $U^\triangleright$. Dually, if $V$ is a subset of the Abelian group $G^\wedge$, the set of all $x \in G$ such that $\chi(x) \in \mathbb{T}_+$ is called the *inverse polar* of $V$ and denoted by $V^\triangleleft$. A set $U \subset G$ is said to be *quasi-convex* if $U = U^{\triangleright\triangleleft}$. Keeping in mind Hahn-Banach separation theorems, topological Abelian groups with a basis of quasi-convex neighborhoods of zero (the so-called *locally quasi-convex groups*) are expected to be a convenient generalization of locally convex spaces. (We shall make more concrete this statement below.)

Let $G$ be a topological Abelian group. Denote by $\sigma(G, G^\wedge)$ the coarsest topology for which all characters in $G^\wedge$ are continuous; in other words, the initial topology on $G$ with respect to all homomorphisms in $G^\wedge$. $\sigma(G, G^\wedge)$, usually known as *Bohr topology* of $G$, is a precompact group topology, and it admits as a subbasis of neighborhoods of zero the family of sets $\{\chi\}^\triangleleft$ with $\chi$ running over $G^\wedge$.

It is common to consider group topologies on the dual group $G^\wedge$. The most usual one is the *compact-open topology* (topology of uniform convergence on compact subsets of $G$), which admits as a basis of neighborhoods at 1 the family of all polars of compact subsets of $G$ (this is also an easy consequence of Proposition (1.1)). We will denote by $G^\wedge_c$ the group $G^\wedge$ equipped with the compact-open topology.

The following computation of the dual of $\mathbb{R}$ is classical; we single it out here for later use. For any fixed real number $t$, let $p_t : \mathbb{R} \to \mathbb{T}$ be the mapping defined by $p_t(\lambda) = p(\lambda t)$ for every $\lambda \in \mathbb{R}$.

PROPOSITION (1.3). *For the additive group $\mathbb{R}$ endowed with its usual topology we have*

$$\mathbb{R}^\wedge = \{p_t : t \in \mathbb{R}\}$$

*and the mapping $t \mapsto p_t$ is a topological group isomorphism between $\mathbb{R}$ and $\mathbb{R}^\wedge_c$.*

For an elementary proof of this result see for instance [21], Lem. 21.5, Prop. 21.8(a).

In what follows we will be mainly working in the context of a real vector space $E$ endowed with a topology $\mathcal{T}$ such that $(E, \mathcal{T})$ is a topological Abelian group with respect to addition (we will abbreviate this to "an additive group topology" in what follows). A real vector space endowed with an additive group topology will be called *a group-topologized vector space* over $\mathbb{R}$.

Let $E$ be a group-topologized vector space over $\mathbb{R}$. We denote by $E^*$ the set of all continuous linear functionals $f : E \to \mathbb{R}$. The set $E^*$ carries a natural vector space structure; this vector space, when equipped with the topology of uniform convergence on all compact subsets of $E$, will be denoted by $E^*_c$.

We want to study the relationship between continuous linear functionals and continuous characters defined on a group-topologized vector space $E$ over $\mathbb{R}$. It is clear that if $f \in E^*$, then $p \circ f \in E^\wedge$.

The following notion arises naturally:

*Definition* (1.4). A group-topologized vector space $E$ over $\mathbb{R}$ is said to have the *character lifting property* if for every continuous character $\chi : E \to \mathbb{T}$ there exists a continuous linear functional $f : E \to \mathbb{R}$ for which $p \circ f = \chi$.

Proposition (1.3) says, in particular, that $\mathbb{R}$ has the character lifting property. The above quoted result from [20] and [14] can be formulated as the fact that any real topological vector space has the character lifting property. This is our Corollary (3.11) below; an alternative proof of this result can be found in [13], (23.32).

We will show that in some cases the following question has a positive answer: let $E$ be a group-topologized vector space over $\mathbb{R}$ with the character lifting property; is $E$ necessarily a topological vector space?

## 2. Group topologies on vector spaces

Recall that a group-topologized vector space $(E, \mathcal{T})$ over $\mathbb{R}$ is said to be a *topological vector space* if the map

$$\begin{array}{ccc} \mathbb{R} \times E & \longrightarrow & E \\ (\lambda, x) & \mapsto & \lambda x \end{array}$$

is continuous when considering the usual topology on $\mathbb{R}$, the topology $\mathcal{T}$ on $E$ and the product topology on $\mathbb{R} \times E$.

Next we weaken this requirement in several different ways. Recall that a subset $U$ of a real vector space $E$ is said to be *balanced* if $[-1, 1]U = U$, and *absorbing* if for every $x \in E$ there exists $\alpha > 0$ such that $x \in \lambda U$ whenever $|\lambda| > \alpha$.

*Definition* (2.1). A group-topologized vector space $(E, \mathcal{T})$ over $\mathbb{R}$ is said to be

(a) *locally absorbing* if every neighborhood of zero in $E$ is absorbing,
(b) *locally balanced* if $E$ has a basis of neighborhoods of zero formed by balanced sets,
(c) a *topological vector group* if for every $\lambda \in \mathbb{R}$, the map

$$\begin{array}{ccc} E & \longrightarrow & E \\ x & \mapsto & \lambda x \end{array}$$

is continuous.

Note that a group-topologized vector space $(E, \mathcal{T})$ over $\mathbb{R}$ is locally absorbing if and only if for every $x \in E$ the mapping $\left[\lambda \in \mathbb{R} \mapsto \lambda x \in E\right]$ is continuous (this simple fact was pointed out in [16]).

Topological vector groups[1] where first explicitly defined and studied in [19], but the condition defining this class had been separately considered before (see e. g. [3], §1).

In the following results and examples we give some information about the relationships among these classes of spaces.

PROPOSITION (2.2) ([3], §1.5, Proposition 4). *Let $(E, \mathcal{T})$ be a group-topologized vector space over $\mathbb{R}$. Then $(E, \mathcal{T})$ is a topological vector space if and only if it is locally balanced and locally absorbing.*

An analogue of Proposition (2.2) is true in the complex case, but not for a general locally compact valued base field. For simplicity we consider only the real case in this paper, but most of the results are easily generalizable to complex spaces.

PROPOSITION (2.3). *Let $(E, \mathcal{T})$ be a group-topologized vector space over $\mathbb{R}$. If $(E, \mathcal{T})$ is locally balanced, then $(E, \mathcal{T})$ is a topological vector group.*

*Proof.* We must show that $\frac{1}{\lambda}U$ is a neighborhood of zero for any fixed $\lambda \neq 0$ and $U \in \mathcal{N}_0(E)$.

Fix $n \in \mathbb{N}$ such that $|\lambda| \leq n$. Since $\mathcal{T}$ is a group topology, there exists a neighborhood of zero $W$ such that $W + \overset{n}{\ldots} + W \subset U$. In particular $W \subset \frac{1}{n}U$. We may assume that $U$ is balanced, and thus

$$W \subset \frac{1}{n}U \subset \frac{1}{|\lambda|}U = \frac{1}{\lambda}U$$

so $\frac{1}{\lambda}U$ is a neighborhood of zero, as required. □

PROPOSITION (2.4). *Let $(E, \mathcal{T})$ be a locally absorbing topological vector group. If $(E, \mathcal{T})$ is either metrizable or a Baire space then it is a topological vector space.*

*Proof.* This follows from the fact that a separately continuous biadditive map defined on the product of a Baire group and a metrizable group is continuous ([22], Theorem 11.15). □

[1] Note that the term "vector group" has been used in algebraic Abelian group theory for more general groups than those underlying real vector spaces, and "topological vector group" has been chosen in some references to designate the object we have called "group-topologized vector space".

*Example* (2.5). A nontrivial vector space endowed with the discrete topology is a locally balanced topological vector group which is not locally absorbing.

*Example* (2.6). (This example uses some basic facts about group topologies defined by characters; details not provided here can be found in [6], §2.3 or [4], §3). Let $H$ be a dense subgroup of $\mathbb{R}$. Consider the initial topology on $\mathbb{R}$ with respect to the family of characters $\Gamma_H = \{p_h : h \in H\} \subset \mathbb{R}^\wedge$. We will denote this topology by $\tau_H$. Clearly $\tau_{\mathbb{R}} = \sigma(\mathbb{R}, \mathbb{R}^\wedge)$ is the Bohr topology of $\mathbb{R}$.

(a) $(\mathbb{R}, \tau_H)$ is a Hausdorff precompact noncompact topological group. Moreover, $\tau_H$ is strictly coarser than the usual topology of $\mathbb{R}$.

b) $(\mathbb{R}, \tau_H)$ is a locally absorbing connected group-topologized vector space over $\mathbb{R}$. Moreover, $(\mathbb{R}, \tau_H)$ is not a topological vector space over $\mathbb{R}$.

(c) $(\mathbb{R}, \tau_H)$ is not locally balanced.

(d) $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$ is a locally absorbing precompact topological vector group. Moreover, the scalar multiplication

$$\begin{array}{ccc} \mathbb{R} \times (\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge)) & \longrightarrow & (\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge)) \\ (\lambda, x) & \mapsto & \lambda x \end{array}$$

is sequentially continuous, but not continuous.

(e) If $(\mathbb{R}, \tau_H)$ is a topological vector group, then $H = \mathbb{R}$.

*Proof.* (a). It is well known that $(\mathbb{R}, \tau_H)$ is a topological group. It is Hausdorff because $\Gamma_H$ separates points of $\mathbb{R}$. $(\mathbb{R}, \tau_H)$ is a precompact group, because it is topologically isomorphic to a subgroup of the compact group $\mathbb{T}^H$. Clearly, $\tau_H$ is coarser than the usual topology $\mathcal{T}_u$ of $\mathbb{R}$. Since $\mathbb{R}$ is not precompact, we get that $\tau_H$ is strictly coarser than $\mathcal{T}_u$. (For a proof of this statement which does not use this precompactness argument, see [5].) Assume that $(\mathbb{R}, \tau_H)$ is compact; from this, according to the Open Mapping Theorem (see e. g. [15], Theorem 3) we would deduce that the continuous identity mapping from $\mathbb{R}$ to $(\mathbb{R}, \tau_H)$ is open as well; hence we would get a contradiction: $\mathcal{T}_u = \tau_H$. (However, there exist nontrivial compact topological vector groups, see Remark (5.2).)

(b) Since $\tau_H$ is coarser than the usual topology of $\mathbb{R}$, we deduce that $(\mathbb{R}, \tau_H)$ is locally absorbing and connected. $(\mathbb{R}, \tau_H)$ is not a topological vector space over $\mathbb{R}$ because it is precompact and nontrivial.

(c) $(\mathbb{R}, \tau_H)$ is not locally balanced, because otherwise from the first part of (b) and Proposition (2.2) we would get that $(\mathbb{R}, \tau_H)$ is a topological vector space over $\mathbb{R}$, which would contradict the second part of (b). (Actually the only balanced neighborhood of zero in $(\mathbb{R}, \tau_H)$ is the whole group; this follows easily from the direct proof given in [5] of the fact that any $\tau_H$-neighborhood of zero contains arbitrarily large real numbers).

d) Fix $\lambda \in \mathbb{R}$. We need to show that the mapping $x \mapsto m_\lambda(x) := \lambda x$ is $(\sigma(\mathbb{R}, \mathbb{R}^\wedge), \sigma(\mathbb{R}, \mathbb{R}^\wedge))$-continuous. This is true because $p_h \circ m_\lambda = p_{h\lambda}$ for any $h \in \mathbb{R}$. Consequently, the first part is proved. The scalar multiplication is sequentially continuous because $\mathbb{R}$ and $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$ have the same convergent sequences (this is a classical result, see for instance [9]). It is not continuous because $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$ is not a topological vector space over $\mathbb{R}$.

(e) Fix $\lambda \in \mathbb{R}$. Assume that the mapping $x \mapsto m_\lambda(x) = \lambda x$ is $(\tau_H, \tau_H)$-continuous. This implies that for every $h \in H$ the composition $p_h \circ m_\lambda = p_{h\lambda}$ is $\tau_H$-continuous. From this (see [4], Theor. 3.7 or [6], Theor. 2.3.4.) we get that $p_{h\lambda} \in \Gamma_H$ for every $h \in H$. Hence, $h\lambda \in H$ for every $h \in H$. If $(\mathbb{R}, \tau_H)$ is a topological vector group, then for every $\lambda \in \mathbb{R}$ the mapping $x \mapsto m_\lambda(x) = \lambda x$ is $(\tau_H, \tau_H)$-continuous. Consequently we get $h\lambda \in H, \forall h \in H, \forall \lambda \in \mathbb{R}$ so $H = \mathbb{R}$. $\square$

*Example* (2.7). Let $E$ be as in Example (2.5), and $F = (\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$ (see Example (2.6)). The space $E \times F$, with the product topology, is a Hausdorff topological vector group which is neither locally absorbing nor locally balanced.

In the remaining of this section we will discuss local convexity and local quasi-convexity of additive group topologies on a real vector space.

*Definition* (2.8) (cf. [2], 9.1). A group-topologized vector space over $\mathbb{R}$ is said to be a *locally convex vector group* if it has a basis of neighborhoods of zero formed by convex sets.

Locally convex vector groups are clearly locally balanced. From Proposition (2.3) it easily follows that any locally convex vector group is a (locally convex) topological vector group.

Locally convex vector groups were first defined in [19]. They became an interesting object of study in view of their good duality properties (see [11]) and the fundamental role they play in the theory of nuclear groups ([1], [2], [8]).

It is known (see [2], 2.4) that a topological vector space is locally quasi-convex as a group if and only if it is a locally convex topological vector space. It is *not* true that a topological vector group is locally quasi-convex if and only if it is a locally convex vector group; indeed, it is easy to show that the non locally balanced topological vector group $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$ (see Example (2.6)) is locally quasi-convex.

In spite of this, something can be said about the relationship between local convexity and local quasi-convexity in topological vector group setting. We first state a Lemma:

LEMMA (2.9) (cf. [7], §18). *Let $(E, \mathcal{T})$ be a group-topologized vector space over $\mathbb{R}$. For every balanced subset $B$ of $E$, the convex envelope* $\operatorname{co} B$ *of $B$ is contained in $B^{\triangleright\triangleleft}$.*

*Proof.* Fix $b_1, \dots, b_n$ in $B$ and $t_1, \dots, t_n$ nonnegative numbers such that $\sum_{k=1}^n t_k = 1$. We must show that

$$b = \sum_{k=1}^n t_k b_k \in B^{\triangleright\triangleleft},$$

i.e. that $\chi(b) \in \mathbb{T}_+$ for every $\chi \in B^\triangleright$. Fix such a $\chi$ and define the characters

$$\chi_k \colon \mathbb{R} \to \mathbb{T}, \quad \chi_k(\alpha) = \chi(\alpha b_k), \quad k \in \{1, \dots, n\}.$$

Since $B$ is balanced and $\chi \in B^\triangleright$, we have $\chi_k([-1, 1]) \subset \mathbb{T}_+$ for every $k \in \{1, \dots, n\}$. Hence the characters $\chi_k$ are continuous (see Proposition (1.2)). By Proposition (1.3), for every $k \in \{1, \dots, n\}$ there exists $\lambda_k \in \mathbb{R}$ with $\chi_k(\alpha) =$

$p(\lambda_k\alpha)$ for every $\alpha \in \mathbb{R}$. From the inclusion $\lambda_k[-1,1] \subset [-1/4,1/4]+\mathbb{Z}$ we deduce $\lambda_k \in [-1/4,1/4]$, for every $k \in \{1,\dots,n\}$. Hence

$$\chi(b) = \chi(\sum_{k=1}^{n} t_k b_k) = \prod_{k=1}^{n} \chi(t_k b_k) \prod_{k=1}^{n} \chi_k(t_k) = p(\sum_{k=1}^{n} \lambda_k t_k) \in \mathbb{T}_+$$

since $\sum_{k=1}^{n} \lambda_k t_k \in \sum_{k=1}^{n} t_k[-1/4,1/4] \subset [-1/4,1/4]$. □

PROPOSITION (2.10). *Let $E$ be a group-topologized vector space over $\mathbb{R}$.*
(a) *(Banaszczyk) If $E$ is a locally convex vector group, then it is locally quasi-convex.*
(b) *If $E$ is locally balanced and locally quasi-convex, then it is a locally convex vector group.*

*Proof.* (a). This statement is contained in the proof of [2], Theorem 15.7; it explicitly coincides with [1], Corollary 9.9.

(b). Fix an arbitrary $U \in \mathcal{N}_0(E)$; we shall find a convex zero neighborhood contained in $U$. Find a quasi-convex $W_1 \in \mathcal{N}_0(E)$ contained in $U$. Next find a balanced $W_2 \in \mathcal{N}_0(E)$ contained in $W_1$. By Lemma (2.9), $\operatorname{co} W_2 \subset W_2^{\triangleright\triangleleft}$. Now, since $W_1$ is quasi-convex and $W_2 \subset W_1$, it is immediate that $W_2^{\triangleright\triangleleft} \subset W_1^{\triangleright\triangleleft} = W_1$. Hence $\operatorname{co} W_2$ is a convex neighborhood of zero contained in $U$. □

## 3. Lifting continuous characters

In this section we shall consider conditions under which a continuous character on a group-topologized vector space $E$ over $\mathbb{R}$ can be lifted (i.e. written as $p \circ f$) to a continuous linear functional $f : E \to \mathbb{R}$. First we prove some related algebraic results:

LEMMA (3.1). *Let $E$ be a real vector space and $f$, $g$ group homomorphisms from $E$ to $\mathbb{R}$ with $p \circ f = p \circ g$. Then $f = g$.*

*Proof.* (Suggested by the referee.) Let $h = f - g$. Then $p \circ h = 0$, so $h(E) \subset \ker p = \mathbb{Z}$. Since $h(E)$ is divisible, this is possible only if $h = 0$, i. e. $f = g$. □

PROPOSITION (3.2). *Let $E$ be a real vector space, $\chi : E \to \mathbb{T}$ a character. The following are equivalent:*
(a) *There exists a linear functional $f : E \to \mathbb{R}$ such that $p \circ f = \chi$*
(b) *For every $x \in E$ there exists $\mu_x \in \mathbb{R}$ with*

$$\chi(\lambda x) = p(\lambda\mu_x) \quad \forall\lambda \in \mathbb{R}\,.$$

(c) *For every $x \in E$, the function $[\lambda \in \mathbb{R} \mapsto \chi_x(\lambda) := \chi(\lambda x) \in \mathbb{T}]$ is continuous.*

*Proof.* (a)⇒(c): Fix $x \in E$; the function $\chi_x : \mathbb{R} \to \mathbb{T}$ is continuous because for any $\lambda \in \mathbb{R}$, $\chi_x(\lambda) = \chi(\lambda x) = p(f(\lambda x)) = p(\lambda f(x))$ and $p : \mathbb{R} \to \mathbb{T}$ is continuous.

(c)⇒(b): Fix $x \in E$; since $\chi : E \to \mathbb{T}$ is a character, the function $\chi_x : \mathbb{R} \to \mathbb{T}$ is a continuous character and (b) is true by Proposition (1.3).

(b)⇒(a): By Lemma (3.1), $\mu_x$ is unique. Define thus $f(x) = \mu_x$. To show that $f$ is linear simply use that $\chi(\lambda(x+y)) = p(\lambda(\mu_x+\mu_y))$ for every $\lambda \in \mathbb{R}$ and every $x, y \in E$ and that $\chi(\lambda(\eta x)) = p(\lambda\eta\mu_x)$ for every $\lambda, \eta \in \mathbb{R}$ and $x \in E$. □

*Remark* (3.3). The following "real character version" of Proposition (3.2) is true: Let $E$ be a real vector space, $\chi\colon E \to \mathbb{T}$ a character. The following are equivalent:

(i). There exists a group homomorphism $f\colon E \to \mathbb{R}$ such that $p \circ f = \chi$.

(ii). For every $x \in E$, the function $[\lambda \in \mathbb{Q} \mapsto \chi_x(\lambda) := \chi(\lambda x) \in \mathbb{T}]$ is continuous.

Using this result one can find a character $\chi\colon E \to \mathbb{T}$ for which (i) does not hold, even for $E = \mathbb{R}$.

COROLLARY (3.4). *Let $(E, \mathcal{T})$ be a locally absorbing group-topologized vector space over $\mathbb{R}$. Then for any $\mathcal{T}$-continuous character $\chi\colon E \to \mathbb{T}$ there exists a unique (not necessarily $\mathcal{T}$-continuous) linear functional $f\colon E \to \mathbb{R}$ such that $p \circ f = \chi$.*

*Proof.* The existence of such an $f$ follows from implication (c)⇒(a) of Proposition (3.2); the uniqueness follows from Lemma (3.1). □

THEOREM (3.5). *Let $(E, \mathcal{T})$ be a group-topologized vector space over $\mathbb{R}$ and $\chi\colon E \to \mathbb{T}$ be a $\mathcal{T}$-continuous character. The following are equivalent:*

(a) *There exists a $\mathcal{T}$-continuous linear functional $f\colon E \to \mathbb{R}$ with $p \circ f = \chi$.*

(b) (b1) *For every $x \in E$, $\big[\lambda \in \mathbb{R} \mapsto \chi(\lambda x) \in \mathbb{T}\big]$ is continuous, and*

(b2) *The mapping $\big[(\lambda, x) \in \mathbb{R} \times (E, \mathcal{T}) \mapsto \chi(\lambda x) \in \mathbb{T}\big]$ is "locally not onto" at $(0, 0)$, i.e. there exist $U \in \mathcal{N}_0(E, \mathcal{T})$ and $\delta > 0$ with $\chi([-\delta, \delta]U) \neq \mathbb{T}$.*

(c) *The mapping $\big[(\lambda, x) \in \mathbb{R} \times (E, \mathcal{T}) \mapsto \chi(\lambda x) \in \mathbb{T}\big]$ is continuous.*

*Proof.* (a)⇒(c): Note that $\chi(\lambda x) = p(f(\lambda x)) = p(\lambda f(x))$ and thus the biadditive map $\big[(\lambda, x) \mapsto \chi(\lambda x)\big]$ may be written as a composition of continuous maps:

$$(\lambda, x) \in \mathbb{R} \times E \mapsto (\lambda, f(x)) \in \mathbb{R} \times \mathbb{R} \mapsto \lambda f(x) \in \mathbb{R} \mapsto p(\lambda f(x)) \in \mathbb{T}\,.$$

(c)⇒(b) is clear.

(b)⇒(a): Since (b1) is satisfied, by Proposition (3.2) there exists a linear functional $f\colon E \to \mathbb{R}$ such that $p \circ f = \chi$. For $f$ to be continuous it is sufficient its being bounded on a neighborhood of zero. Since (b2) is satisfied there exist $\delta > 0$, $U \in \mathcal{N}_0(E)$ and $\theta \in \mathbb{R}$ such that $p(\theta) \notin \chi([-\delta, \delta]U) = p([-\delta, \delta]f(U))$. Consequently, the sets $[-\delta, \delta]f(U)$ and $\theta + \mathbb{Z}$ are disjoint. From this, since $[-\delta, \delta]f(U)$ is balanced, we get that $[-\delta, \delta]f(U)$ is bounded. Hence, the set $f(U)$ is bounded as well. □

*Example* (3.6). There are nonliftable characters $\chi \in E^\wedge$ for which $[\lambda \in \mathbb{R} \mapsto \chi(\lambda x) \in \mathbb{T}]$ $(x \in E)$ and $[x \in E \mapsto \chi(\lambda x) \in \mathbb{T}]$ $(\lambda \in \mathbb{R})$ are both continuous, e. g. any nontrivial continuous character defined on the locally absorbing topological vector group $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$ (see Example (2.6)).

*Example* (3.7). There are nonliftable characters $\chi \in E^\wedge$ for which the maps $[x \in E \mapsto \chi(\lambda x) \in \mathbb{T}]$ $(\lambda \in \mathbb{R})$ are continuous and $[(\lambda, x) \in \mathbb{R} \times E \mapsto \chi(\lambda x) \in \mathbb{T}]$ is continuous at zero: actually, if $E = \mathbb{R}$ with the discrete topology, the map $[(\lambda, x) \in \mathbb{R} \times E \mapsto \lambda x \in E]$ is continuous at zero; and $E$ has nonliftable characters (those which are discontinuous with respect to the ordinary topology; see e. g. [13], 25.6).

Next we collect some particular cases of additional properties of the space $E$ which are useful in lifting characters:

PROPOSITION (3.8). *Let $(E, \mathcal{T})$ be a locally balanced group-topologized vector space over $\mathbb{R}$ and let $\chi\colon E \to \mathbb{T}$ be a $\mathcal{T}$-continuous character. The following are equivalent:*

(i) *There exists a $\mathcal{T}$-continuous linear functional $f\colon E \to \mathbb{R}$ with $p \circ f = \chi$.*

(ii) *For every $x \in E$, the function $\left[\lambda \in \mathbb{R} \mapsto \chi(\lambda x) \in \mathbb{T}\right]$ is continuous.*

*Proof.* In view of Theorem (3.5), only the implication (ii)⇒(i) needs a proof. This implication follows from (b)⇒(a) in Theorem (3.5). Actually, since $\chi$ satisfies hypothesis (b1) in this result, we need to show only that $\chi$ satisfies (b2), too. Since $\chi$ is continuous and $E$ is locally balanced, there exists a balanced neighborhood of zero $U$ in $E$ for which $\chi(U) \subset \mathbb{T}_+ \neq \mathbb{T}$. Then $\chi([-1, 1]U) = \chi(U) \neq \mathbb{T}$. □

PROPOSITION (3.9). *Let $E$ be a group-topologized vector space over $\mathbb{R}$. If $E$ is either metrizable or Baire and $\chi \in E^\wedge$ is such that the biadditive map $[(\lambda, x) \in \mathbb{R} \times E \mapsto \chi(\lambda x) \in \mathbb{T}]$ is separately continuous, then there exists a continuous linear functional $f\colon E \to \mathbb{R}$ with $p \circ f = \chi$.*

*Proof.* This is another consequence of [22], Theorem 11.15; in view of the implication (c)⇒(a) of Theorem (3.5). □

Recall that $E$ is said to have the character lifting property if every continuous character on $E$ can be lifted to a continuous linear functional. Using this terminology, the equivalence (a)⇔(c) of Theorem (3.5) can be written as follows:

THEOREM (3.10). *Let $E$ be a group-topologized vector space over $\mathbb{R}$. The following conditions are equivalent:*

(i) *$E$ has the character lifting property.*

(ii) *The mapping*

$$\begin{array}{ccc} \mathbb{R} \times E & \longrightarrow & (E, \sigma(E, E^\wedge)) \\ (\lambda, x) & \mapsto & \lambda x \end{array}$$

*is jointly continuous.*

COROLLARY (3.11) ([14], [20]). *If $E$ is a real topological vector space, then $E$ has the character lifting property.*

*Proof.* As $E$ is a topological vector space and the topology $\sigma(E, E^\wedge)$ is coarser than the topology of $E$, the conclusion follows from (ii)⇒(i) in Theorem (3.10). □

In connection with Corollary (3.11) it is natural to pose the following question: *If $(E, \mathcal{T})$ is a group-topologized vector space over $\mathbb{R}$ which has the character lifting property, is then $(E, \mathcal{T})$ a topological vector space?*

If we do not impose on $E$ some extra conditions making the dual group rich enough, the answer to the above question is negative, as the following example shows.

*Example* (3.12). Let $\mathcal{T}$ be a group topology on $\mathbb{R}$ such that $(\mathbb{R}, \mathcal{T})^\wedge = \{1\}$ (such a topology exists, even a metrizable one; see [17]). Then the group-topologized vector space $(\mathbb{R}, \mathcal{T})$ trivially has the character lifting property, but it is not a topological vector space.

There is an important class of topological vector groups for which having the character lifting property and being a topological vector space are equivalent conditions:

THEOREM (3.13). *Let $E$ be a locally convex vector group over $\mathbb{R}$. If $E$ has the character lifting property then $E$ is a locally convex topological vector space.*

*Proof.* By Proposition (2.2), it suffices to show that $E$ is locally absorbing. Suppose this is not so. Since $E$ is locally convex, we can find an $x \in E$ and a balanced convex $U \in \mathcal{N}_0(E)$ such that $x$ not absorbed by $U$. Since $U$ is balanced and convex, we have $\operatorname{sp} U = \bigcup_{t\in\mathbb{R}} tU$. Hence,

$$\operatorname{sp}\{x\} \cap \operatorname{sp} U = \{0\}\,.$$

Let $E = \operatorname{sp}\{x\} \oplus \operatorname{sp} U \oplus F$ be an algebraic decomposition of $E$. Fix some discontinuous character $\kappa\colon \mathbb{R} \to \mathbb{T}$. Define now

$$\chi\colon E \to \mathbb{T}, \quad \chi(\mu x + u + f) = \kappa(\mu)$$

for every $u \in \operatorname{sp} U$ and $f \in F$. $\chi$ is clearly a character; moreover, it is continuous since $\chi(U) = \{1\}$ (Proposition (1.2)). Let us prove that $\chi$ is not liftable. Suppose that there exists $f \in E^*$ with $p \circ f = \chi$. Then, for every $\mu \in \mathbb{R}$,

$$\kappa(\mu) = \chi(\mu x) = p(f(\mu x)) = p(\mu f(x))$$

and $\kappa$ would be continuous, as a composition of continuous maps:

$$\mu \in \mathbb{R} \mapsto \mu f(x) \in \mathbb{R} \mapsto p(\mu f(x)) \in \mathbb{T}\,.$$

□

*Remark* (3.14). This proof actually shows that the following variant of Theorem (3.13) is true: Let $E$ be a group-topologized vector space over $\mathbb{R}$. Suppose that $E$ has a basis $\mathcal{B}$ of neighborhoods of zero such that for every $U \in \mathcal{B}$, $U$ is balanced and $\operatorname{sp} U = \bigcup_{t\in\mathbb{R}} tU$. If for every continuous character $\chi$ on $E$ there is a linear functional $f\colon E \to \mathbb{R}$ with $p \circ f = \chi$, then $E$ is a topological vector space.

COROLLARY (3.15). *A real vector space endowed with the discrete topology has not the character lifting property unless it is trivial.*

An application of Proposition (2.10) yields the following reformulation of Theorem (3.13) :

PROPOSITION (3.16). *Let $E$ be a group-topologized vector space over $\mathbb{R}$. If $E$ is locally balanced, locally quasi-convex and has the character lifting property, then it is a locally convex topological vector space.*

The following question arises: Is it possible to eliminate the assumption of local balancedness from Proposition (3.16)?

## 4. Nickolas' theorem and liftable characters

The following result can be extracted from the proof of the main theorem in [18]:

THEOREM (4.1). *Let $G$ be a topological Abelian group which is a k-space. Then for every character $\chi$ in the path-connected component of $1$ in $G_c^\wedge$ there exists a continuous homomorphism $f\colon G \to \mathbb{R}$ with $\chi = p \circ f$.*

THEOREM (4.2). *Let $E$ be a group-topologized vector space over $\mathbb{R}$ and $\chi\colon E \to \mathbb{R}$ a continuous character. Define for each $t \in \mathbb{R}$ the character $[x \in E \mapsto \chi_t(x) := \chi(tx)]$. Consider the following properties:*

(i) *There exists a continuous linear functional $f\colon E \to \mathbb{R}$ with $p \circ f = \chi$.*

(ii) *For every $x \in E$ the function $[t \in \mathbb{R} \mapsto \chi(tx)]$ is continuous and moreover, the mapping $[t \in \mathbb{R} \mapsto \chi_t \in E_c^\wedge]$ is continuous.*

(iii) *For every $x \in E$ the character $[t \in \mathbb{R} \mapsto \chi(tx)]$ is continuous and moreover, $\chi$ is in the path-connected component of $1$ in $E_c^\wedge$.*
*Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii) and, if $E$ is a k-space, (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii).*

*Proof.* (i)$\Rightarrow$(ii): Suppose that there exists $f \in E^*$ with $p \circ f = \chi$. Then clearly $[t \in \mathbb{R} \mapsto \chi(tx)]$ is continuous. On the other hand, in order to prove that $[t \in \mathbb{R} \mapsto \chi_t \in E_c^\wedge]$ is continuous we must show that for every compact $K \subset E$ there exists $\varepsilon > 0$ with

$$|t| \leq \varepsilon,\ x \in K \Rightarrow \chi(tx) \in \mathbb{T}_+ .$$

Given $t \in \mathbb{R}$ and $x \in E$, we have $\chi(tx) = p(tf(x))$. Since $K$ is compact, $f(K)$ is a bounded subset of $\mathbb{R}$. From this the existence of such an $\varepsilon > 0$ follows at once.

(ii)$\Rightarrow$(iii) is immediate.

(iii)$\Rightarrow$(i) if $E$ is a k-space: By Proposition (3.2), $\chi$ can be lifted to a linear functional $f$. By Theorem (4.1), $\chi$ can be lifted to a continuous homomorphism $g\colon E \to \mathbb{R}$. By Lemma (3.1), $f = g$. □

*Example* (4.3). Consider the locally balanced topological vector group $E = (\mathbb{R}, \mathcal{T}_d)$, where $\mathcal{T}_d$ denotes the discrete topology. $E$ is metrizable, hence a k-space. By Hamel's classical result ([12]), there exist many nonhomogeneous group homomorphisms $f\colon \mathbb{R} \to \mathbb{R}$. Let $f$ be such a homomorphism. The character $p \circ f\colon E \to \mathbb{T}$ can be lifted to a continuous homomorphism $f\colon E \to \mathbb{R}$ but not to a linear functional. This shows that the condition concerning continuity of $[t \in \mathbb{R} \mapsto \chi(tx)]$ $(x \in E)$ cannot be dispensed with in (iii).

*Example* (4.4). Let $E$ be the topological vector group $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$ (Example (2.6)). Consider the (nonliftable) continuous character $\chi$ on $E$ defined by $\chi(x) = p(x)$. The space $E_c^\wedge$ is topologically isomorphic with $\mathbb{R}$ with the usual topology; this is a consequence of the classical Glicksberg theorem ([10]), and the fact that $\mathbb{R}$ is topologically self-dual (Proposition (1.3) above). This shows that (ii)$\Rightarrow$(i) does not hold in general.

## 5. The space of liftable characters

Let $E$ be a group-topologized vector space over $\mathbb{R}$.

For $E$ two kinds of dual objects can be considered: the set $E^*$ of all continuous linear functionals $f\colon E \to \mathbb{R}$ and the set $E^\wedge$ of all continuous characters $\chi\colon E \to \mathbb{T}$.

The set $E^*$ is a vector space over $\mathbb{R}$ and it is usually called *the dual space* of $E$. The dual space $E^*$ endowed with the compact-open topology will be denoted by $E^*_c$. (A basis of neighborhoods of zero for the compact-open topology on $E^*$ is given by the sets $K^\circ = \{f \in E^* : f(K) \subset [-1, 1]\}$, where $K$ is an arbitrary compact subset of $E$.) Note that $E^*_c$ is a Hausdorff locally convex topological vector space.

As we have already mentioned, the set $E^\wedge$ is an Abelian group under pointwise product of characters; it is called *the dual group* of $E$. We denote by $E^\wedge_c$ the dual group $E^\wedge$ endowed with the compact-open topology. Note that $E^\wedge_c$ is a Hausdorff locally quasi-convex topological Abelian group.

Suppose that $E$ is a topological vector group over $\mathbb{R}$. Then the Abelian group $E^\wedge$ carries a natural vector space structure; namely, if $\chi \in E^\wedge$ and $\lambda \in \mathbb{R}$ we may define $\lambda\chi\colon E \to \mathbb{T}$ by the expression $(\lambda\chi)(x) := \chi(\lambda x)$, $x \in E$. Since the mapping $x \mapsto \lambda x$ is continuous, we get that $\lambda\chi \in E^\wedge$.

PROPOSITION (5.1). *Let $E$ be a topological vector group over $\mathbb{R}$. Then $E^\wedge_c$ is also a topological vector group over $\mathbb{R}$.*

*Proof.* $E^\wedge_c$ is a topological vector group since $\lambda K$ is a compact subset of $E$ for any $\lambda \in \mathbb{R}$ and any compact $K \subset E$. □

*Remark* (5.2). Let $E \neq \{0\}$ be a vector space over $\mathbb{R}$ endowed with the discrete topology. Then $E^\wedge_c$ presents an example of a nontrivial **compact** topological vector group over $\mathbb{R}$. Indeed, $E^\wedge_c$ is a topological vector group over $\mathbb{R}$ by Proposition (5.1); $E^\wedge_c$ is nontrivial and compact, because it is the Pontryagin dual group of a nontrivial discrete Abelian group $E$.

Let us introduce a mapping $P\colon E^* \to E^\wedge$ by the following equality: $P(f) = p \circ f$, $f \in E^*$. Clearly, $P(E^*)$ is the subgroup of $E^\wedge$ formed by those characters which are liftable to continuous linear forms. We denote by $P(E^*)_c$ the set $P(E^*)$ endowed with compact-open topology.

PROPOSITION (5.3). *Let $E$ be a topological vector group. Then*

(a) $P\colon E^*_c \to E^\wedge_c$ *is an injective continuous linear mapping;*

(b) $P(E^*)_c$ *is a locally absorbing topological vector group.*

*Proof.* (a) is straightforward.

(b): Since by Proposition (5.1) $E^\wedge_c$ is a topological vector group, its subspace $P(E^*)_c$ is a topological vector group as well.

To show that $P(E^*)_c$ is locally absorbing, for any $\chi \in P(E^*)$ and any compact $K \subset E$ we must find $\varepsilon > 0$ with $\chi([-\varepsilon, \varepsilon]K) \subset \mathbb{T}_+$. It suffices to choose $\varepsilon > 0$ with $f(K) \subset \left[-\frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right]$, where $f \in E^*$ is the lifting of $\chi$. □

For a topological vector group $E$ the mapping $P\colon E_c^* \to E_c^\wedge$ may fail to be a topological embedding (this is for instance the case if $E$ is $\mathbb{R}$ endowed with the discrete topology). Below we present two non-pathological cases.

PROPOSITION (5.4). *Let $E$ be a hemicompact topological vector group. Then $P\colon E_c^* \to E_c^\wedge$ is a topological embedding (equivalently, $P(E^*)_c$ is a topological vector space).*

*Proof.* Since $E$ is hemicompact, $E_c^\wedge$ (and thus, $P(E^*)_c$) is a metrizable group. By Proposition (5.3), the biadditive map $[(\lambda, \chi) \in \mathbb{R} \times P(E^*)_c \mapsto \lambda\chi \in P(E^*)_c]$ is separately continuous. By [22], Theorem 11.15; it is jointly continuous. □

PROPOSITION (5.5). *Let $E$ be a topological vector group. Suppose that every compact subset of $E$ is contained in some connected (or balanced) compact set. Then $P\colon E_c^* \to E_c^\wedge$ is a topological embedding (equivalently, $P(E^*)_c$ is a topological vector space).*

*Proof.* We know that $P\colon E_c^* \to E_c^\wedge$ is a continuous monomorphism. To show that it is open onto its image we need to prove the following: For every compact $K \subset E$ there exists a compact $S \subset E$ such that $P(K^\circ) \supset S^\triangleright \cap P(E^*)$; equivalently,

$$f \in E^*,\ f(S) \subset [-1/4, 1/4] + \mathbb{Z} \ \Rightarrow\ f(K) \subset [-1, 1].$$

Given a compact $K \subset E$, choose a compact and connected (resp. a compact and balanced) $S$ which contains $K$. Since $f(S)$ is a connected set for any $f \in E^*$, if $f(S) \subset [-1/4, 1, 4] + \mathbb{Z}$ then in fact $f(S) \subset [-1/4, 1, 4]$ and thus $f(K) \subset [-1/4, 1, 4]$. □

COROLLARY (5.6) ([20]). *If $E$ is a topological vector space, then $P$ is a topological isomorphism between $E_c^*$ and $E_c^\wedge$.*

*Proof.* This follows from Proposition (5.5) together with Corollary (3.11). □

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