

# Cyclic coverings and Seshadri constants on smooth surfaces.

Luis Fuentes García

## Abstract

We study the Seshadri constants of cyclic coverings of smooth surfaces. The existence of an automorphism on these surfaces can be used to produce Seshadri exceptional curves. We apply this method to  $n$ -cyclic coverings of the projective plane. When  $2 \leq n \leq 9$ , explicit values are obtained. We relate this problem with the Nagata conjecture.

**MSC (2000):** Primary 14E20; secondary, 14C20.

**Key Words:** Cyclic coverings, Seshadri constants, Nagata conjecture.

## 1 Introduction.

The Seshadri constants were introduced by Demailly in [8]. If  $X$  is a smooth surface, the Seshadri constant of an ample line bundle  $L$  at  $x \in X$  is defined as:

$$\epsilon(L, x) = \inf_{C \ni x} \left\{ \frac{L \cdot C}{\text{mult}_x(C)} \mid C \text{ irreducible curve through } x \right\}.$$

A well known upper bound for the Seshadri constants on surfaces is  $\epsilon(L, x) \leq \sqrt{L^2}$ . When the constant does not reach this bound, there is a curve with high multiplicity at  $x$  such that

$$\epsilon(L, x) = \frac{C \cdot L}{\text{mult}_x(C)}.$$

These curves are called Seshadri exceptional curves (see [9]).

General bounds for the Seshadri constants on surfaces are given in [2], [12] or [17]. An interesting open problem is their irrationality. It seems that surfaces with irrational Seshadri constants must exist. However, the explicit known values are always rational. They were computed for simple abelian surfaces by Th. Bauer and T. Szemberg (see [1]); Ch. Schultz gave values for Seshadri constants on products of two elliptic curves (see [16]); I obtained the Seshadri constants on elliptic ruled surfaces (see [11]). Note that, in all these cases, the

construction of the Seshadri exceptional curves providing the constants is based in the same idea. The existence of an involution on these surfaces allows to construct invariant curves with high multiplicity at the fixed points.

In this paper we use this idea to study Seshadri constants on  $n$ -cyclic coverings of smooth surfaces:

$$\pi : X \longrightarrow Y.$$

These surfaces have an automorphism of order  $n$ . We search Seshadri exceptional curves on invariant linear systems. With this method, we will obtain results about the Seshadri constants of line bundles  $\pi^*L$  of  $X$  at points on the ramification divisor.

We extend a result of Steffens about Seshadri constants on surfaces with Picard number 1 (see [17]). We prove:

**Theorem 1.1** *Let  $\pi : X \longrightarrow Y$  be a  $n$ -cyclic branched covering of a smooth surface  $Y$  with  $\rho(Y) = 1$ . Let  $L$  be an ample generator of  $NS(Y)$ . Then, if  $\eta$  is a very general point on  $X$ :*

$$\left[ \sqrt{(\pi^*L)^2} \right] \leq \epsilon(\pi^*L, \eta) \leq \sqrt{(\pi^*L)^2}.$$

*In particular, if  $\sqrt{(\pi^*L)^2}$  is an integer, then  $\epsilon(\pi^*L, \eta) = \sqrt{(\pi^*L)^2}$ .*

We also apply the method to study  $n$ -cyclic coverings of the projective plane. When  $2 \leq n \leq 9$  we obtain explicit values of the Seshadri constant of  $\pi^*\mathcal{O}_{P^2}(1)$  at a very general point  $x$  on the ramification divisor (Theorem 4.6):

| $n$                                      | 2 | 3   | 4 | 5 | 6    | 7    | 8     | 9 |
|--|---|-----|---|---|------|------|-------|---|
| $\epsilon(\pi^*\mathcal{O}_{P^2}(1), x)$ | 1 | 3/2 | 2 | 2 | 12/5 | 21/8 | 48/17 | 3 |

Finally, we see the relation between the study of the Seshadri constant of  $\pi^*\mathcal{O}_{P^2}(1)$  and the Nagata conjecture. In particular, the Seshadri exceptional curves for  $\pi^*\mathcal{O}_{P^2}(1)$  are given by the pullback of curves in  $\mathbf{P}^2$  passing through  $n$  infinitely near points with prescribed order. This take us to establish the following conjecture:

**Conjecture 1.2** *Let  $\pi : X \longrightarrow \mathbf{P}^2$  be a  $n$ -cyclic covering of the projective plane, with generic branch divisor of degree  $nb$ . If  $n > 9$  then the Seshadri constant of  $\pi^*\mathcal{O}_{P^2}(1)$  at a very general point  $\eta$  is maximal:*

$$\epsilon(\pi^*\mathcal{O}_{P^2}(1), \eta) = \sqrt{n}.$$

Note, that the veracity of this conjecture will provide examples of polarized surfaces with irrational Seshadri constants.

We refer to [2] for a systematic study of the main properties of the Seshadri constants on surfaces.

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## 2 Cyclic coverings.

Firstly, we recall some well known facts about cyclic coverings (see [3]). Let  $Y$  be a smooth surface and let  $\mathcal{O}_Y(M)$  be a line bundle on  $Y$ . Consider a smooth reduced divisor  $B \sim nM$  on  $Y$ . Then we have a  $n$ -cyclic covering:

$$\pi : X \longrightarrow Y$$

with branch divisor  $B$ . Since  $B$  is smooth and reduced,  $X$  is a smooth surface. Let  $R$  be the reduced divisor  $\pi^{-1}(B)$  (the ramification divisor). It holds:

1.  $\mathcal{O}_X(R) = \pi^* \mathcal{O}_Y(M)$ .
2.  $\pi^* B = nR$ .
3.  $\mathcal{O}_X(K_X) = \pi^*(\mathcal{O}_Y(K_Y + (n-1)M))$ .

On the other hand, there is an induced automorphism of order  $n$ :

$$\sigma : X \longrightarrow X.$$

The fixed points of  $\sigma$  are exactly the points of the ramification divisor  $R$ . Moreover, we have automorphisms:

$$\bar{\sigma} : H^0(X, \pi^* L) \longrightarrow H^0(X, \pi^* L), \quad \text{where } L \text{ is a line bundle on } Y.$$

The spaces of eigenvectors of these automorphisms are given by the following decomposition:

$$H^0(X, \pi^* L) \cong \bigoplus_{k=0}^{n-1} H^0(Y, L - kM).$$

In fact, if  $D \sim \pi^* L$  is a  $\sigma$ -invariant divisor then  $D = E + kR$ , where  $E \sim \pi^*(L - kM)$ . Since we are interested on irreducible divisors, we will study invariant divisors associated to the eigenvalue 1.

We will denote by  $H^0(X, \pi^* L)_1$  the space of sections associated to the eigenvalue 1. Let us consider a point  $x$  at the ramification divisor  $R$ . We study the existence of divisors in  $H^0(X, \pi^* L)_1$  passing through  $x$  with given multiplicity.

Let us take local coordinates  $(u, v)$  such that  $v = 0$  is the local equation of  $R$  and  $u = 0$  corresponds to an invariant divisor passing through  $x$ . The automorphism  $\sigma$  have the following local expression:

$$\sigma(u, v) = (u, \theta v),$$

where  $\theta$  is a primitive  $n$ -root of unity.

Let  $f(u, v) = 0$  be a local equation of a divisor in  $H^0(X, \pi^* L)_1$ . It must verify:

$$f(u, v) = f(u, \theta v).$$

From this:

$$\frac{\partial f}{\partial u^i \partial v^j}(0, 0) = \theta^j \frac{\partial f}{\partial u^i \partial v^j}(0, 0)$$

and then:

$$\frac{\partial f}{\partial u^i \partial v^j}(0, 0) = 0, \quad \text{when } j \neq 0 \pmod n.$$

Therefore, we deduce the following lemma:

**Lemma 2.1** *With the previous notation, the number of conditions on a divisor in  $H^0(X, \pi^*L)_1$  to pass through  $x \in R$  with multiplicity at least  $m$  is:*

$$(k+1) \binom{\frac{nk}{2} + r}{r}, \quad \text{where } m = nk + r, \quad 0 \leq r \leq n-1.$$

■

### 3 Cyclic coverings of smooth surfaces with Picard number 1.

In [17], Steffens proved the following result:

**Theorem 3.1** *Let  $X$  be a surface with  $\rho(X) = \text{rk}(NS(X)) = 1$  and let  $L$  be an ample generator of  $NS(X)$ . Let  $\alpha$  be an integer with  $\alpha^2 \leq L^2$ . If  $\eta \in X$  is a very general point, then  $\epsilon(L, \eta) \geq \alpha$ . In particular, if  $\sqrt{L^2}$  is an integer, then  $\epsilon(L, \eta) = \sqrt{L^2}$ .*

The proof is based on two facts. First, the following result of Ein-Lazarsfeld ([10]):

**Lemma 3.2** *Let  $\{C_t \in x_t\}_{t \in \Delta}$  be a 1-parameter family of reduced irreducible curves on a smooth projective surface  $X$ , such that  $\text{mult}_{x_t}(C_t) \geq m$  for all  $t \in \Delta$ . Then:*

$$(C_t)^2 \geq m(m-1).$$

Moreover, because  $\rho(X) = 1$ , he can use that a Seshadri exceptional curve  $C$  is numerically equivalent to  $dL$  for some integer  $d$ .

This result can be generalized to cyclic coverings of surfaces with Picard number 1. Note that a cyclic covering  $\pi : X \rightarrow Y$  verifies  $\rho(X) \geq \rho(Y)$ .

**Theorem 3.3** *Let  $\pi : X \rightarrow Y$  be a  $n$ -cyclic branched covering of a smooth surface  $Y$  with  $\rho(Y) = 1$ . Let  $L$  be an ample generator of  $NS(Y)$ . Then, if  $\eta$  is a very general point on  $X$ :*

$$\left\lceil \sqrt{(\pi^*L)^2} \right\rceil \leq \epsilon(\pi^*L, \eta) \leq \sqrt{(\pi^*L)^2}.$$

In particular, if  $\sqrt{(\pi^*L)^2}$  is an integer, then  $\epsilon(\pi^*L, \eta) = \sqrt{(\pi^*L)^2}$ .

**Proof:** We will prove the theorem for a very general point  $x$  of the ramification divisor  $R$ . Since the Seshadri constant is lower semi-continuous (see [13]), this implies the result for a very general point  $\eta \in X$ .

Let us suppose that the Seshadri constant of  $\pi^*L$  at  $x$  does not reach the expected value. Then there is an irreducible Seshadri exceptional curve  $C \subset X$  providing the Seshadri constant:

$$\epsilon(\pi^*L, x) = \frac{C \cdot \pi^*L}{m}, \quad \text{with } m = \text{mult}_x(C).$$

Moreover, we know that:

$$C^2 \geq m(m-1).$$

If we want to apply the idea of Steffens, the curve  $C$  should be a multiple of  $\pi^*L$ . But this is not true in general. However, we can solve this problem in the following way.

Let  $\sigma : X \rightarrow X$  be the induced automorphism on  $X$ . There is an integer  $l$  dividing  $n$  such that the divisor

$$D \equiv C + \sigma(C) + \sigma^2(C) + \dots + \sigma^{l-1}(C)$$

is invariant for the involution. Since  $\rho(Y) = 1$ , there is an integer  $j$  such that,  $D \equiv j\pi^*L$ . Furthermore, each curve  $\sigma^p(C)$  passes through  $x$  with multiplicity  $m$ . From this:

$$D^2 = lC^2 + 2 \sum_{p \neq q} \sigma^p(C) \cdot \sigma^q(C) \geq lm(m-1) + l(l-1)m^2 = lm(lm-1),$$

where  $lm = \text{mult}_x(D)$ . Finally, the divisor  $D$  provides the Seshadri constant:

$$\frac{D \cdot \pi^*L}{\text{mult}_x(D)} = \frac{lC \cdot \pi^*L}{lm} = \epsilon(\pi^*L, x).$$

Now, we can apply the arguments of Steffens to conclude the result. ■

**Remark 3.4** *The argument used in this proof give us an interesting consequence. In order to compute the Seshadri constant of line bundles  $\pi^*L$  at points in the ramification divisor, we only have to consider divisors that are invariant for the involution  $\sigma$ . Note, that this does not depend on the Picard number of  $Y$ .*

## 4 Cyclic coverings of the projective plane.

We will work with a  $n$ -cyclic covering of  $\mathbf{P}^2$ :

$$\pi : X \rightarrow \mathbf{P}^2$$

with branch divisor  $B \sim nbL$  and  $L = \mathcal{O}_{P^2}(1)$ . We are interested on the Seshadri constant of  $\pi^*L$ . A direct application of the Theorem 3.3 gives:

**Theorem 4.1** *Let  $\pi : X \rightarrow \mathbf{P}^2$  be an  $n$ -cyclic covering of  $\mathbf{P}^2$  and  $L = \mathcal{O}_{P^2}(1)$ . If  $\eta$  is a very general point on  $X$  then:*

$$[\sqrt{n}] \leq \epsilon(\pi^*L, \eta) \leq \sqrt{n}.$$

In particular, if  $\sqrt{n}$  is an integer,  $\epsilon(\pi^*L, \eta) = \sqrt{n}$ . ■

We can obtain a refinement of this result. Let us consider the linear system  $|d\pi^*L|$ . Let  $x$  be a point on the ramification divisor  $R$ . We will use the Lemma 2.1 to find divisors in  $|d\pi^*L|_1$  with high multiplicity  $m$  at  $x$ .

A Seshadri exceptional curve  $D \sim d\pi^*L$  passing through  $x$  must verify:

$$D^2 \leq m^2, \quad \text{or equivalently,} \quad d^2n \leq m^2. \quad (1)$$

The dimension of  $H^0(X, d\pi^*L)_1$  is:

$$h^0(\mathcal{O}_{P^2}(d)) = \binom{d+2}{2}.$$

This dimension must be greater than the number of conditions on a divisor  $D$  to pass through  $x$  with multiplicity  $m$ . Applying the Lemma 2.1, this means:

$$\binom{d+2}{2} > (k+1)\left(\frac{nk}{2} + r\right) \iff \binom{d+2}{2} - 1 \geq \left(\frac{m-r}{n} + 1\right) \left(\frac{m+r}{2}\right) \quad (2)$$

where  $m = nk + r$ , with  $0 \leq r < n$ . From this:

$$d^2 + 3d \geq \frac{m^2 - r^2}{n} + m + r \stackrel{(1)}{\implies} 3d \geq m + r - \frac{r^2}{n} \geq m.$$

Taking squares and applying the inequality (1):

$$9d^2 \geq m^2 \geq nd^2 \implies n \leq 9.$$

Thus, if  $n > 9$  we can not get the desired divisor  $D$ . On the other hand, if  $2 \leq n \leq 9$  we can expect to find values of  $m$  and  $d$  satisfying the inequalities (1) and (2). Explicitly, we obtain:

| n | d | m  | $h^0(\mathcal{O}_{P^2}(d))$ | conditions |
|---|---|----|-----------------------------|------------|
| 2 | 1 | 2  | 3                           | 2          |
| 3 | 1 | 2  | 3                           | 2          |
| 4 | 1 | 2  | 3                           | 2          |
| 5 | 2 | 5  | 6                           | 5          |
| 6 | 2 | 5  | 6                           | 5          |
| 7 | 3 | 8  | 10                          | 9          |
| 8 | 6 | 17 | 28                          | 27         |
| 9 | 3 | 9  | 10                          | 9          |

Table 1: Seshadri exceptional divisors for  $\pi^*\mathcal{O}_{P^2}(1)$ .

Now, let us see that these exceptional (possibly reduced) divisors are optimal. We will prove two lemmas for bounding the multiplicity of any Seshadri exceptional divisor for  $\pi^*L$ .

**Lemma 4.2** *Let  $x$  be a generic point on the ramification divisor. Let  $C \sim d\pi^*L$  be a Seshadri exceptional divisor for  $\pi^*L$  at  $x$ . Then  $\text{mult}_x(C) < h^0(\mathcal{O}_{P^2}(d))$ .*

**Proof:** Let  $m = \text{mult}_x(C)$ . Note that:

$$(dL \cdot B)_{\pi(x)} = (C \cdot R)_x \geq m.$$

This means that there is a plane curve  $C'$  of degree  $d$  meeting  $B$  with multiplicity at least  $m$  at  $\pi(x)$ . Let us consider the Veronese map of degree  $d$ :

$$v_d : \mathbf{P}^2 \longrightarrow \mathbf{P}^{h^0(\mathcal{O}_{P^2}(d))-1}.$$

Now, the curve  $C'$  corresponds to a hyperplane meeting  $v_d(B)$  at  $y = v_d(\pi(x))$  with multiplicity at least  $m$ . When  $y$  is a generic point, this multiplicity is upper bounded by the dimension of the ambient space. ■

**Lemma 4.3** *When  $n \neq 8$ , the exceptional divisors related on the Table 1 provide the Seshadri constant of  $\pi^*L$  at a very general point on the ramification divisor.*

**Proof:** If  $n = 4, 9$  the result follows from the Theorem 4.1.

Let  $D_n$  be the Seshadri exceptional divisor described on the table 1. Let  $C$  be a Seshadri exceptional curve providing the Seshadri constant of  $\pi^*L$  at a very general point  $x$  on the ramification divisor  $R$ . With the same argument of the proof of the Theorem 3.3, we can construct a divisor  $D \sim j\pi^*L$  invariant by the involution and providing the Seshadri constant of  $\pi^*L$ . The number  $j$  is the smaller integer such that  $j\pi^*L$  contains the curve  $C$ . Thus, since  $D_n$  and  $D$  are Seshadri exceptional divisors multiple of  $\pi^*L$ , it holds  $j \leq d$ . Moreover, if  $m = \text{mult}_x(D)$ :

$$m^2 \geq D^2 \geq m(m-1), \quad \text{where } D^2 = nj^2. \quad (3)$$

Now, we use this inequality and the Lemma 4.2 to discard the cases which do not appear on the Table 1.

1. If  $n = 2, 3$ , then  $j \leq 1$ . The unique possibility is  $m = 2$ .
2. If  $n = 5, 6$ , then  $j \leq 2$ . If  $j = 1$ , by the Lemma 4.2,  $m < 3$  and the inequality (3) fails. If  $j = 2$ , necessarily  $m = 5$ .
3. If  $n = 7$ , then  $j \leq 3$ . If  $j = 1$  or  $j = 2$ , then  $m < 3$  or  $m < 6$  respectively. In both cases the inequality (3) fails. If  $j = 3$ , necessarily  $m = 8$ . ■

**Remark 4.4** *Let us try to apply the same arguments when  $n = 8$ . In this case  $j \leq 6$ . With the inequality (3) and Lemma 4.2, we can discard the cases  $j = 1, 2, 4, 5$ . The problem appears when  $j = 3$  and  $m = 9$ . Let us see that we can eliminate this possibility if we work with a generic branch divisor.*

**Lemma 4.5** *When  $n = 8$  and the branch divisor  $B$  is a generic plane curve of degree  $8b$ , the Seshadri constant of  $\pi^*L$  at a very general point  $x$  on the ramification divisor is given by the divisor described on the table 1.*

**Proof:** We must discard the case  $j = 3$  and  $m = 9$ . The existence of a divisor  $D \sim 3\pi^*L$  passing through  $x$  with multiplicity 9 implies the existence of a cubic curve with multiplicity 2 at  $x$  and meeting the branch divisor with multiplicity at least 9. Let us see that this curve does not exist. Since we suppose that  $B$  is generic, it is sufficient to find an example.

Let us consider affine coordinates  $(x, y)$ . Let us take the curve  $B$  given by the equation:

$$y = x^{8b} + x^4 + x^2.$$

Now, with a direct computation, we can check that there are not any curve of degree 3 with a singular point at  $(0, 0)$  and meeting  $B$  with multiplicity 9 at the same point. ■

As a consequence of the previous discussion we have proved the following Theorem:

**Theorem 4.6** *Let  $\pi : X \rightarrow \mathbf{P}^2$  be a  $n$ -cyclic covering of  $\mathbf{P}^2$ , with  $2 \leq n \leq 9$  and a generic branch divisor of degree  $bn$ . Let  $L \sim \mathcal{O}_{\mathbf{P}^2}(1)$ . Then:*

1. *If  $x$  is a very general point on the ramification divisor:*

|                       |   |     |   |   |      |      |       |   |
|-----------------------|---|-----|---|---|------|------|-------|---|
| $n$                   | 2 | 3   | 4 | 5 | 6    | 7    | 8     | 9 |
| $\epsilon(\pi^*L, x)$ | 1 | 3/2 | 2 | 2 | 12/5 | 21/8 | 48/17 | 3 |

2. *If  $\eta$  is a very general point on  $X$ :*

$$\epsilon(\pi^*L, x) \leq \epsilon(\pi^*L, \eta) \leq \sqrt{n}$$



In particular, the global Seshadri constant  $\epsilon(\pi^*L)$  is rational. ■

**Remark 4.7** When  $n = 2$  the surface  $X$  is a double covering of the projective plane. The Theorem states that  $\epsilon(\pi^*L, x) = 1$ , when  $x$  is a very general point on the ramification divisor. Let  $2b$  be the degree of the branch divisor. It is well known that  $X$  is one of the following surfaces:

1. If  $b = 1$ ,  $X \cong \mathbf{P}^1 \times \mathbf{P}^1$ . The divisor  $\pi^*L$  is of type  $(1,1)$  on  $X$ . Its Seshadri constant at any point is 1.
2. If  $b = 2$ , then  $X$  is a Del Pezzo surface. In particular,  $X$  is the blowing up of  $\mathbf{P}^2$  at 7 general points. The divisor  $\pi^*L$  corresponds to the anticanonical divisor  $-K_X$  of  $X$ . If  $x$  is a generic point on  $X$  then:

$$\epsilon(\pi^*L, x) = \frac{4}{3}.$$

However, when  $x$  is on the ramification divisor  $\epsilon(\pi^*L, x) = 1$ .

3. If  $b = 3$ , then  $X$  is a K3 surface. We prove that  $\epsilon(\pi^*L, x) = 1$  at a very general point  $x$  on the ramification divisor. The same result is a direct consequence of the Example 2.3 of [4]. There, it is proved that  $d\pi^*L$  generates exactly  $d$ -jets at  $x$ . The relation between the Seshadri constants and generation of jets provides the desired value of  $\epsilon(\pi^*L, x)$  (see Proposition 1.1 of [2]).
4. If  $b > 3$ , then  $X$  is a surface of general type. By a result of Buium (see [5]), when the branch divisor is generic,  $X$  has Picard number 1. Thus, we conclude that the global Seshadri constant of any line bundle on a general double cover of the projective plane is rational.

## 5 Relation with the Nagata conjecture.

Consider the problem of the existence of plane algebraic curves of given degree and with singularities of prescribed order in points in general position (see [18] for a survey on this topic). The Nagata conjecture says:

**Conjecture 5.1 (Nagata conjecture)** Let  $P_1, \dots, P_n$  be  $n \geq 10$  be general points in  $\mathbf{P}^2$  and let  $k_1, \dots, k_n$  be fixed non-negative integers. If  $C \subset \mathbf{P}^2$  is a curve of degree  $d$  such that  $\text{mult}_{P_i}(C) \geq k_i$  then:

$$d \geq \frac{1}{\sqrt{n}} \sum_{i=1}^n k_i.$$

This can be reformulated in the language of Seshadri constants. If  $(Y, L)$  is polarized variety,  $\epsilon(Y; n)$  denotes the multiple Seshadri constant of  $L$  at  $n$  general points.

**Conjecture 5.2 (Nagata conjecture via Seshadri constants)** *If  $n \geq 9$ , the Seshadri constant  $\epsilon(\mathcal{O}_{P^2}(1); n)$  is maximal, that is,*

$$\epsilon(\mathcal{O}_{P^2}(1); n) = \frac{1}{\sqrt{n}}.$$

Let us see the relation between our result and the Nagata conjecture. Let  $\pi : X \rightarrow \mathbf{P}^2$  be a  $n$ -cyclic covering of the projective plane. Let  $L = \mathcal{O}_{P^2}(1)$ . Let  $x$  be a generic point on the ramification divisor. Let us suppose that the Seshadri constant  $\epsilon(\pi^*L, x)$  does not reach the expected value. In this case, we have seen that the Seshadri constant is given by a divisor  $D = \pi^*C$ , with  $C \sim \mathcal{O}_{P^2}(d)$ . The multiplicity of  $D$  at  $x$  determines the infinitesimal behavior of  $C$  at  $\pi(x)$ . This can be expressed in the language of clusters in the following way (see [6], [7]).

Let  $m = \text{mult}_x(D) = nk + r$ , with  $0 \leq r < n$ . Let  $S_1 = \mathbf{P}^2$  and  $P_1 = \pi(x)$ . Let  $S_i$  be the blowing up of  $S_{i-1}$  at  $P_{i-1}$ , with exceptional divisor  $E_i$ . Let  $P_i$  be the intersection of the strict transform of the branch divisor and  $E_i$ . Then, the curve  $C$  passes through the cluster  $(P_1, \dots, P_n)$  with multiplicities  $(k+1, \dots, k+1, k, \dots, k)$ . Moreover:

$$\frac{C \cdot L}{r(k+1) + (n-r)k} < \frac{1}{\sqrt{n}} \iff \frac{D \cdot \pi^*L}{m} < \sqrt{n}.$$

Thus, the problem of finding a Seshadri exceptional curves for  $\pi^*L$  on  $X$  is a special case of the general problem of finding exceptional curves on  $\mathbf{P}^2$  (see [14], [15]).

If  $2 \leq n \leq 9$  the exceptional curves described on the Table 1 corresponds to the the exceptional curves providing the  $n$ -tuple Seshadri constants on  $\mathbf{P}^2$  (see Example 2.4 of [18]). When  $n > 9$  the Nagata conjecture is not solved, except if  $n$  is a square. This is agree with the Theorem 4.1.

As a consequence of this discussion, we establish the following conjecture:

**Conjecture 5.3** *Let  $\pi : X \rightarrow \mathbf{P}^2$  be a  $n$ -cyclic covering of the projective plane, with generic branch divisor of degree  $nb$  and  $n > 9$ . Then the Seshadri constant of  $\pi^*\mathcal{O}_{P^2}(1)$  at a very general point  $\eta$  is maximal:*

$$\epsilon(\pi^*\mathcal{O}_{P^2}(1), \eta) = \sqrt{n}.$$

**Remark 5.4** *The points  $P_1, \dots, P_n$  of the cluster depends on the branch divisor. Therefore, the condition of “general points” on the Nagata conjecture is replaced here for the condition of “generic branch divisor”.*

**Remark 5.5** *When  $n$  is not a square, this conjecture will provide an example of surface with irrational Seshadri constant.*

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**E-mail:** lfuentes@udc.es

Luis Fuentes García.

Departamento de Métodos Matemáticos y Representación.

E.T.S. de Ingenieros de Caminos, Canales y Puertos.

Universidad de A Coruña. Campus de Elviña. 15192 A Coruña (SPAIN)