NUCLEAR AND GP-NUCLEAR GROUPS

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Dedicated to Professor S. Császár on the ocassion of his 75th. birthday

ABSTRACT. We define GP-nuclear groups as topological Abelian groups for which the groups of summable and absolutely summable sequences are the same algebraically and topologically. It is shown that in the metrizable case only the algebraic coincidence of the mentioned groups is needed for GP-nuclearity. Some permanence properties of the class of GP-nuclear groups are obtained. Our final result asserts that nuclear groups in the sense of Banaszczyk are GP-nuclear. The validity of the converse assertion remains open.

1. Introduction

The theory of nuclear locally convex spaces is now an important part of Functional Analysis. The notion of a nuclear space has been introduced by Grothendieck in the framework of his theory of topological tensor products. Now the following definition of nuclear spaces is commonly accepted (see [15, 17, 19]):

Definition 1.1. A locally convex space E is said to be *nuclear* if for any absolutely convex neighborhood of zero U in E there exists another absolutely convex neighborhood of zero $V \subset U$ such that the canonical mapping of E_V into E_U is nuclear. Here E_U denotes the Banach space associated with U.

Several characterizations of nuclear spaces are known. In the definition, instead of the ideal of nuclear operators many others can be used (see [15, p. 482]). Grothendieck established already the following important internal criterion: a metrizable locally convex space is nuclear if and only if every summable sequence in it is also absolutely summable ([17, p. 69]). Metrizability cannot be dropped from this assertion. Pietsch ([17, p. 73]) characterized nuclear spaces as those locally convex spaces for which appropriately topologized spaces of summable and absolutely summable sequences are the same algebraically and topologically. We shall refer to this result as *Grothendieck-Pietsch theorem.*

Mitiagin's theorem describes nuclear spaces in terms of Kolmogorov diameters of neighborhoods of zero, as follows: a locally convex space E is nuclear if and only if for any absolutely convex neighborhood of zero U in E there exists another absolutely convex neighborhood of zero $V \subset U$ such that $\lim_{k\to\infty} k^p d_k(V, U) = 0$ for some, resp. for each, $p > 0$. Here $d_k(X, Y)$ denotes the kth Kolmogorov diameter of X with respect to Y (see [15, p. 485]).

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In his interesting cycle of papers ([3, 4, 5]) Banaszczyk demonstrated that closed subgroups and Hausdorff quotient groups of nuclear Fréchet spaces have many nice properties, e. g., they are Pontryagin reflexive, any closed subgroup of a nuclear Fréchet space is weakly closed, etc. The specific methods and tools needed for the proofs of these results allow him to give in [2] the following definition of nuclear groups, which is based on Mitiagin's theorem:

Definition 1.2. A Hausdorff Abelian topological group G is called *nuclear* when it satisfies the following condition:

Given an arbitrary neighborhood of zero U in G, $c > 0$ and $m \in \mathbb{N}$ there exist: a vector space E, two symmetric and convex subsets X, Y of E with $d_k(X, Y) \leq ck^{-m}$ $\forall k \in \mathbb{N}$, a subgroup K of E and a homomorphism $\phi : K \to G$, such that $\phi(K \cap X)$ is a neighborhood of zero in G and $\phi(K \cap Y) \subset U$.

The above definition is not internal and in a glance may seem to be complicated, but it leads to many significant results. As it has already been shown in [2], the class of nuclear groups contains locally compact Abelian groups and nuclear locally convex spaces and it satisfies all the permanence properties which the class of nuclear spaces has. Many important theorems known for nuclear locally convex spaces have been generalized for nuclear groups (see Chapter 4 of [2], [6] or [7]). An extensive study of nuclear groups can be found in [1] and [14].

The starting point for the present note is [6], in which it is shown that in an arbitrary nuclear group any summable sequence is absolutely summable. Consequently, the necessity part of Grothendieck's criterion is obtained. Concerning the sufficiency part the following is written: "If every (weakly) summable family in a [metrizable] locally convex space is absolutely summable, then the space is nuclear ([17, 4.2.4]). It seems conceivable that *locally convex space* may be replaced by *locally quasi-convex group*".

We are attempting to study the question of validity of Grothendieck-Pietsch theorem for nuclear groups in full strength. This means the following: For a given topological Abelian group G let us consider the groups $\ell_1(G)$ and $\ell_1\{G\}$ of all summable and absolutely summable sequences in G. We equip these groups with the naturally defined topologies in such a way that when G is a locally convex space they coincide with the known ones, which are called respectively ε-topology and π-topology in [17]. The above mentioned result of [6] says that if G is a nuclear group, $\ell_1(G) = \ell_1\{G\}$ as sets, i. e., algebraically. Two questions arise:

Question 1. Let G be an arbitrary nuclear group. Are then the topological groups $\ell_1(G)$ and $\ell_1\{G\}$ also the same topologically?

Question 2. Let G be an arbitrary (locally quasi-convex) topological Abelian group such that the topological groups $\ell_1(G)$ and $\ell_1\{G\}$ are the same algebraically and topologically. Is then G nuclear?

To underline the importance of the problem, we propose the following

Definition 1.3. We say that a topological Abelian group G is *Grothendieck-Pietsch nuclear,* or simply *GP-nuclear,* if $\ell_1(G) = \ell_1\{G\}$ algebraically and topologically.

The notion of GP-nuclear group may have also an independent interest, since its definition is formulated by using only tools of group theory.

The paper is organized as follows:

In Section 2 we collect notations and prove some auxiliary lemmas.

In Section 3 we introduce the groups of summable and absolutely summable sequences, and equip them with convenient topologies. The question of topologization of these groups is relatively new. As far as we know, the topologization of the group of [super]summable sequences has been considered only in [11, p. 102]. The question of topologization of the group of absolutely summable sequences seems not to have been considered until now.

In Section 4 GP-nuclear groups are defined and studied. It is observed that the class of GP-nuclear groups is closed under formation of Cartesian products, completions and arbitrary subgroups. When G is metrizable, then for GP-nuclearity of G the algebraic coincidence of $\ell_1(G)$ and $\ell_1\{G\}$ suffices (Theorem 4.8). Our final result (Theorem 4.10) asserts that nuclear groups (in Banaszczyk's sense) are also GP-nuclear, i. e., Question 1 is answered affirmatively. Our proof uses the already cited result of [6], our Theorem 4.8 and a result from [1] which states that any nuclear group is isomorphic with a subgroup of a product of metrizable nuclear groups. The question whether or not any GP-nuclear group is nuclear (i. e., Question 2) remains open.

In Section 5 we show that the groups of summable and absolutely summable sequences and the topologies we have introduced on them coincide in the case of locally convex spaces with traditional ones. It is also proved that the additive group of a metrizable topological vector space E is GP-nuclear if and only if E is a nuclear locally convex space.

2. Auxiliary notations and facts

Notations $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ will have the usual meaning. $\mathbb{T}$ will denote the multiplicative group of complex numbers with modulus one, endowed with the topology induced by that of $\mathbb{C}$.

Unless it is explicitly stated otherwise, throughout the paper G will denote a topological Abelian group with the group operation $+$, the neutral element 0 and with the topology τ_G. By $G^\wedge$ we shall denote its topological dual group. Recall that $G^\wedge$ consists of all continuous characters $\varphi : G \to \mathbb{T}$. $G^\wedge$ is a multiplicative Abelian group; in advance, no topology is defined on it. As in [2], we shall denote by $\mathcal{N}_0(G)$ the family of all neighborhoods of zero in G.

For a nonempty subset $V \subset G$, $V^\triangleright$ will denote the polar of V, i. e.

$$V^\triangleright = \{\varphi \in G^\wedge \ : \ \mathrm{Re}\varphi(x) \geq 0 \ \forall x \in V\}.$$

Also, for a nonempty subset $B \subset G^\wedge$ we shall denote by ${}^\triangleleft B$ the inverse polar of B, i. e.

$${}^\triangleleft B = \{x \in G \ : \ \mathrm{Re}\varphi(x) \geq 0 \ \forall \varphi \in B\}.$$

A subset $V \subset G$ is said to be *quasi-convex* if ${}^\triangleleft(V^\triangleright) = V$. Note that any quasi-convex subset of G is closed and contains zero.

For any nonempty subset $V \subset G$ we define

$$\|x\|_V = \sup_{\varphi \in V^\triangleright} |1 - \varphi(x)|, \quad x \in G.$$

It is easy to verify that $\|\cdot\|_V$ is a quasinorm, i. e. it is subadditive and symmetric. Using this notation we can say that a subset $V \subset G$ is quasi-convex if and only if $V = \{x \in G : \|x\|_V \leq \sqrt{2}\}$. Observe that for a given subset $V \subset G$ the quasinorm $\|\cdot\|_V$ is continuous if and only if $V^{\triangleright}$ is equicontinuous. In particular, it is continuous for any $V \in \mathcal{N}_0(G)$.

G is said to be *locally quasi-convex* if it has a fundamental system of quasi-convex neighborhoods of zero. Observe that G is locally quasi-convex if and only if the system of quasinorms $\{\|\cdot\|_V : V \in \mathcal{N}_0(G)\}$ generates the topology of G. We shall not discuss in detail the notion of local quasi-convexity. Now it is commonly used; see [1], [2] or [10]. Note also that Hausdorff locally quasi-convex groups are dually separated and that any product of locally quasi-convex groups is also locally quasi-convex. The additive topological Abelian group of a real topological vector space E is locally quasi-convex if and only if E is a locally convex space.

The following simple assertion implies the local quasi-convexity of $\mathbb{R}$ and $\mathbb{T}$ and clarifies the meaning of the functional $||\cdot||_V$ in these cases:

Lemma 2.1. (a) *Let $G = \mathbb{T}$ and $n \in \mathbb{N}$. Put*

$$V_n = \{\exp(2\pi i\theta) : \theta \in [-\frac{1}{4n}, \frac{1}{4n}]\} = \{t \in \mathbb{T} : |1-t| \leq 2\sin\frac{\pi}{4n}\}.$$

Then

$$||t||_{V_n} = \max_{k\in\{1,\dots,n\}} |1-t^k| \quad \forall t \in \mathbb{T} \tag{2.1}$$

and

$$||t||_{V_1} = |1-t|, \quad ||t||_{V_n} \leq n|1-t| \quad \forall t \in \mathbb{T}. \tag{2.2}$$

Moreover, V_n is quasi-convex in $\mathbb{T}$, i. e.,

$$t \in \mathbb{T}, \quad Re(t^k) \geq 0 \;\; \forall k \in \{1,\dots,n\} \Longrightarrow |1-t| \leq 2\sin\frac{\pi}{4n}. \tag{2.3}$$

(b) *Let $G = \mathbb{R}$ and $r \in \mathbb{R}$, $r > 0$. Put $V_r = [-r, r]$. Then*

$$\|x\|_{V_r} = \begin{cases} 2\sin\frac{\pi}{4r}|x| & \text{if } |x| \leq 2r \\ 2 & \text{if } |x| \geq 2r \end{cases} \tag{2.4}$$

and

$$x \in V_r \Rightarrow \frac{1}{r}|x| \leq \frac{1}{\sqrt{2}}||x||_{V_r}. \tag{2.5}$$

Moreover, V_r is quasi-convex in $\mathbb{R}$.

Proof. (a) We have $\mathbb{T}^\wedge = \mathbb{Z}$. Using this identification, it is easy to observe that $V_n^{\triangleright} = \{-n,\dots,-1,0,1,\dots,n\}$. This implies (2.1). (2.2) follows at once from (2.1). To prove (2.3), fix $t \in \mathbb{T}$ satisfying the required conditions. We can suppose that $\mathrm{Im} t \geq 0$. Then we have $t = \exp(i\theta)$, $\theta \in [0, \frac{\pi}{2}]$. Since $\mathrm{Re}(t^k) \geq 0$, we have $k\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] + 2\pi\mathbb{Z} \quad \forall k \in \{1,\dots,n\}$. From this we can derive by induction that $n\theta \in [0, \frac{\pi}{2}]$, and (2.3) is proved.

(b) We have $\mathbb{R}^\wedge = \mathbb{R}$, in the sense that any $\varphi \in \mathbb{R}^\wedge$ has the form $\varphi(x) = \exp(2\pi i\lambda x)$ for some $\lambda \in \mathbb{R}$. Using this, it is easy to prove that $V_r^\triangleright = V_{\frac{1}{4r}}$; (2.4) follows from this. (2.5) follows from (2.4) via the inequality $\xi \leq \frac{\pi}{2\sqrt{2}} \sin\xi$, valid for $\xi \in [0, \frac{\pi}{4}]$. □

We shall need the following Lemma about the geometry of $\mathbb{T}$:

Lemma 2.2. *Fix any $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in \mathbb{T}$.*

(a)

$$\max_{\varepsilon_1,\ldots,\varepsilon_n \in \{-1,0,1\}} |1 - \prod_{k=1}^{n} t_k^{\varepsilon_k}| \leq \sqrt{2} \Longrightarrow \sum_{k=1}^{n} |1 - t_k| \leq \frac{\pi}{2\sqrt{2}} \max_{\varepsilon_1,\ldots,\varepsilon_n \in \{-1,0,1\}} |1 - \prod_{k=1}^{n} t_k^{\varepsilon_k}|.$$

(b)

$$\max_{\Delta \subset \{1,\ldots,n\}} |1 - \prod_{k\in\Delta} t_k| \leq \frac{\sqrt{2}}{2} \Longrightarrow \sum_{k=1}^{n} |1 - t_k| \leq \frac{\pi}{\sqrt{2}} \max_{\Delta \subset \{1,\ldots,n\}} |1 - \prod_{k\in\Delta} t_k|.$$

Proof. To prove (a), we can suppose that $\operatorname{Im} t_k \geq 0$ for every $k \in \{1, \ldots, n\}$. In this case we shall show that

$$\sum_{k=1}^{n} |1 - t_k| \leq \frac{\pi}{2\sqrt{2}} |1 - \prod_{k=1}^{n} t_k|. \tag{2.6}$$

We have $t_k = \exp(i\theta_k)$, $\theta_k \in [0, \frac{\pi}{2}]$ for $k \in \{1, \ldots, n\}$. Now our assumption means that for any $\Delta \subset \{1, \ldots, n\}$, $\sum_{k\in\Delta} \theta_k \in [-\frac{\pi}{2}, \frac{\pi}{2}] + 2\pi\mathbb{Z}$. From this we can derive by induction that $\theta := \sum_{k=1}^{n} \theta_k \in [0, \frac{\pi}{2}]$. Finally, using the inequality $\xi \leq \frac{\pi}{2\sqrt{2}} \sin\xi$, valid for $\xi \in [0, \frac{\pi}{4}]$, we obtain

$$\sum_{k=1}^{n} |1 - t_k| = 2\sum_{k=1}^{n} \sin\frac{\theta_k}{2} \leq \sum_{k=1}^{n} \theta_k = 2 \cdot \frac{1}{2}\theta \leq 2 \cdot \frac{\pi}{2\sqrt{2}} \sin\frac{\theta}{2} = \frac{\pi}{2\sqrt{2}} |1 - \prod_{k=1}^{n} t_k|,$$

and (2.6) is proved.

(b) follows from (a) in a standard way. □

Remark 2.3. (b) improves, in the case of $t_k \in \mathbb{T}$, a lemma given in [9, VIII.17]. From the statement in (b) only the relation

$$t \in \mathbb{T},\ Re(t^k) \geq 0\ \forall k \in \{1, \ldots, n\} \Longrightarrow |1 - t| \leq \frac{\pi}{2\sqrt{2}n} |1 - t^n| (\leq \frac{\pi}{2n})$$

can be deduced, which is less accurate than the relation (2.3) in Lemma 2.1(a).

Let G be an Abelian group and U a nonempty subset of it. Let us introduce the functional $k_U : G \to [0, 1]$ as follows (cf. [20, p. 496]):

$$k_U(x) = \sup\{\frac{1}{n} : n \in \mathbb{N},\ x + \overset{n}{\cdots} + x \notin U\} \quad \forall x \in G.$$

As usual, we agree that $\sup\emptyset = 0$.

Below we shall use this functional to define the notion of absolute summability in a group G. A closely related functional $(\cdot/U)$ defined only on U has been considered by Kaplan in [16, p. 650], and afterwards a similar functional has been defined in [2, p. 8].

We call k_U the *Kaplan functional* of U. The Kaplan functional plays the role of the Minkowski functional in the case of groups. Some properties of this functional are collected in the following Lemma:

Lemma 2.4. *Let G be an Abelian group and V, U nonempty subsets of G.*
(a) $U = \{x \in G : k_U(x) < 1\}$.
(b) *If $V \subset U$ then $k_U \leq k_V$.*
(c) *If $V + V \subset U$, then*

$$k_U(x+y) \leq \max\{k_V(x),\, k_V(y)\} \quad \forall x, y \in G.$$

(d) *If $0 \in V$, and $V + V + V \subset U$, then*

$$k_U(x) \leq \frac{1}{2} k_V(x) \quad \forall x \in V.$$

(e) *If $G = \mathbb{T}$ and V_n is as in Lemma 2.1(a), then*

$$k_{V_n}(t) \leq \frac{n}{\sqrt{2}} |1-t| \quad \forall t \in \mathbb{T}$$

and

$$|1-t| \leq \pi k_{V_1}(t) \quad \forall t \in \mathbb{T}.$$

(f) *If $G = \mathbb{R}$ and V_r is as in Lemma 2.1(b), then*

$$k_{V_r}(x) \leq \frac{1}{r}|x| \quad \forall x \in \mathbb{R}$$

and

$$|x| \leq 2k_{V_1}(x) \quad \forall x \in V_1.$$

(g) *If G is a topological Abelian group and $(x_i)_{i \in I}$ is a net in G, then $x_i \to 0$ in G if and only if $k_U(x_i) \to 0$ for any $U \in \mathcal{N}_0(G)$.*

Proof. (a) and (b) are evident.

(c) First suppose that $x + y \notin U$; then either $x \notin V$ or $y \notin V$ and we have $1 = k_U(x+y) = k_V(x)$ or $1 = k_U(x+y) = k_V(y)$. If $x + y \in U$, either $k_U(x+y) = 0$ or $k_U(x+y) = \frac{1}{n+1}$, $n \in \mathbb{N}$. In the first case there is nothing to prove. In the second case, $(x+y) + \overset{n+1}{\cdots} + (x+y) \notin U$, so either $x + \overset{n+1}{\cdots} + x \notin V$ or $y + \overset{n+1}{\cdots} + y \notin V$. We obtain $\max\{k_V(x),\, k_V(y)\} \geq \frac{1}{n+1}$ and the assertion is proved.

(d) If $k_V(x) = 0$, $k_U(x) \leq k_{V+V+V}(x) = 0$ and the inequality is trivial. In other case, $k_V(x) = \frac{1}{n_0+1}$ for some $n_0 \in \mathbb{N}$; in particular, for every $j \in \{1, \ldots, n_0\}$ $x + \overset{j}{\ldots} + x \in V$. Now

$$n \leq 3n_0 \Rightarrow x + \overset{n}{\ldots} + x = x + \overset{n}{\ldots} + x + 0 + \overset{3n_0 - n}{\cdots} + 0 \in V + V + V$$

so

$$\begin{aligned} k_U(x) &\leq k_{V+V+V}(x) = \sup\{\frac{1}{n} : x + \overset{n}{\ldots} + x \notin V + V + V\} \\ &\leq \frac{1}{3n_0+1} = k_V(x)\frac{n_0+1}{3n_0+1} \leq \frac{1}{2}k_V(x). \end{aligned}$$

(e), (f) and (g) are left to the reader. □

Lemma 2.5. *Let G be a topological Abelian group and V a nonempty subset of G. Then*

(a) $\|x\|_V \leq \pi k_V(x) \quad \forall x \in G$.

(b) *If V is quasi-convex, then* $\sqrt{2} k_V(x) \leq \|x\|_V \quad \forall x \in G$.

Proof. (a) is evident if $x \notin V$ (since $\|x\|_V \leq 2 < \pi$ and $k_V(x) = 1$). Suppose now that $x \in V$, and fix arbitrarily $\varphi \in V^{\triangleright}$. If $k_V(x) = 0$ then $x + \overset{m}{\dots} + x \in V$, and so, $\mathrm{Re}(\varphi(x)^m) \geq 0$ for every $m \in \mathbb{N}$. From (2.3) in Lemma 2.1(a) we deduce that $\varphi(x) = 1$, i. e. $\|x\|_V = 0$. If $k_V(x) = \frac{1}{n+1}$ for some $n \in \mathbb{N}$, then $x + \overset{k}{\dots} + x \in V$, and so, $\mathrm{Re}(varphi(x))^k \geq 0$ for every $k \in \{1, \dots, n\}$. Using again (2.3) in Lemma 2.1(a), we obtain that $|1 - \varphi(x)| \leq \frac{\pi}{2n}$. Hence

$$\|x\|_V \leq \frac{\pi}{2}\frac{1}{n} = \frac{\pi}{2}\frac{n+1}{n} k_V(x) \leq \pi k_V(x).$$

(b) If $x \notin V$, $k_V(x) = 1$ and since V is quasi-convex $\|x\|_V > \sqrt{2}$, so the inequality is fulfilled. Suppose that $x \in V$. If $k_V(x) = 0$ there is nothing to prove. In other case, $k_V(x) = \frac{1}{n+1}$ for some $n \in \mathbb{N}$. By definition of k_V, $x + \overset{n+1}{\dots} + x \notin V$, and since V is quasi-convex there exists $\varphi \in V^{\triangleright}$ such that $|1 - \varphi(x)^{n+1}| > \sqrt{2}$; hence $(n+1)|1 - \varphi(x)| \geq |1 - \varphi(x)^{n+1}| > \sqrt{2}$ and $\|x\|_V \geq \frac{\sqrt{2}}{n+1} = \sqrt{2} k_V(x)$. $\square$

3. Presummability and absolute summability

Let us recall first the notion of a summable family. Let A be an infinite set. The family $\mathcal{F}(A)$ of all finite nonempty subsets of A is a directed set with respect to set-theoretic inclusion $\subset$. If now $(x_\alpha)_{\alpha \in A}$ is any family of elements of an Abelian group G, then the family $\left(\sum_{\alpha \in \Delta} x_\alpha\right)_{\Delta \in \mathcal{F}(A)}$ is a net in G.

A family $(x_\alpha)_{\alpha \in A}$ of elements of a topological Abelian group G is called *summable* if the net $\left(\sum_{\alpha \in \Delta} x_\alpha\right)_{\Delta \in \mathcal{F}(A)}$ is convergent in G. This is the ordinary definition of summability given e. g. in [9, III.37].

If $(x_\alpha)_{\alpha \in A}$ is a summable family in a separated topological Abelian group G, then the (unique) limit of the net $\left(\sum_{\alpha \in \Delta} x_\alpha\right)_{\Delta \in \mathcal{F}(A)}$ is denoted by $\sum_{\alpha \in A} x_\alpha$ and it is called the *sum* of $(x_\alpha)_{\alpha \in A}$.

We shall say that the family $(x_\alpha)_{\alpha \in A}$ is *presummable* if $\left(\sum_{\alpha \in \Delta} x_\alpha\right)_{\Delta \in \mathcal{F}(A)}$ is a Cauchy net in (the natural uniformity of) G. Observe that $(x_\alpha)_{\alpha \in A}$ is presummable if and only if the following Cauchy condition of summability is satisfied: for any $V \in \mathcal{N}_0(G)$ there exists a finite subset $\Delta_0 \subset A$ such that whenever $\Delta \in \mathcal{F}(A)$ and $\Delta \cap \Delta_0 = \emptyset$, we have $\sum_{\alpha \in \Delta} x_\alpha \in V$. [1]

Clearly, every summable family is presummable. The converse is true if the group is complete. If G is sequentially complete then any presummable sequence is summable.

For a multiplicative group we shall use the terms "multipliable" and "premultipliable" instead of "summable" and "presummable", and $\prod_{\alpha \in A} x_\alpha$ instead of $\sum_{\alpha \in A} x_\alpha$.

[1] In the literature the terminology is not stated yet; e. g., in [6] and [17] the families satisfying Cauchy condition of summability are named simply "summable families". We decided not to change the meaning of the term "summable family" already accepted in [9] and because of this reason instead of rather long expressions "the family satisfying Cauchy condition of summability" or "unconditionally Cauchy family" we shall use the new term "presummable family".

If $(x_\alpha)_{\alpha\in A}$ is a presummable family and $A_0 \subset A$ is an infinite subset of A, then the new family $(x_\alpha)_{\alpha\in A_0}$ is again presummable. The same may not be true for a summable family in a non-complete group. The family $(x_\alpha)_{\alpha\in A}$ is called *hereditarily summable* ([12]) or *supersummable* ([11, p. 102]) if for any infinite subset $A_0 \subset A$ the family $(x_\alpha)_{\alpha\in A_0}$ is summable in G. In a complete group any summable family is hereditarily summable.

For a given infinite set A and a given topological group G we shall denote by $\ell_1(A,G)$ the set of all presummable families $(x_\alpha)_{\alpha\in A}$ of elements in G. If $A=\mathbb{N}$ we shall write also simply $\ell_1(G)$. Clearly $\ell_1(A,G)$ is a subgroup of G^A and hence itself is a group with respect to the group operation induced from G^A.

We introduce a topology in the group $\ell_1(A,G)$ as follows. Put

$$(V)_A = \{(x_\alpha)_{\alpha\in A} \in \ell_1(A,G) \ : \ \sum_{\alpha\in\Delta} x_\alpha \in V \ \ \forall \Delta \in \mathcal{F}(A)\}, \quad V \in \mathcal{N}_0(G).$$

It is straightforward to check that the system $\{(V)_A \ : \ V \in \mathcal{N}_0(G)\}$ is a fundamental system of neighborhoods of zero for a group topology in $\ell_1(A,G)$. We shall denote this topology with $(\tau_G)_A$. In what follows, we shall always assume that $\ell_1(A,G)$ is equipped with this topology.

A similar topology in the subgroup $\ell(A,G)$ of $\ell_1(A,G)$ consisting of all hereditarily summable families of elements in G has been considered in [11, p. 102], and [12].

Lemma 3.1. *A family $(x_\alpha)_{\alpha\in A}$ in $\mathbb{R}$ is summable if and only if the family $(|x_\alpha|)_{\alpha\in A}$ is summable in $\mathbb{R}$. The group $\ell_1(A,\mathbb{R})$ is also a vector space and its topology can be generated by means of one of the following norms:*

$$(x_\alpha)_{\alpha\in A} \mapsto \sum_{\alpha\in A} |x_\alpha|, \quad (x_\alpha)_{\alpha\in A} \mapsto \sup_{\Delta\in\mathcal{F}(A)} |\sum_{\alpha\in\Delta} x_\alpha|.$$

Moreover,

$$\sup_{\Delta\in\mathcal{F}(A)} |\sum_{\alpha\in\Delta} x_\alpha| \le \sum_{\alpha\in A} |x_\alpha| \le 2 \sup_{\Delta\in\mathcal{F}(A)} |\sum_{\alpha\in\Delta} x_\alpha| \qquad \forall (x_\alpha)_{\alpha\in A} \in \ell_1(A,\mathbb{R}). \tag{3.1}$$

Consequently $\ell_1(A,\mathbb{R})$ is the ordinary Banach space of all summable families in $\mathbb{R}$.

Proof. This is well known. □

Lemma 3.2. *A family $(t_\alpha)_{\alpha\in A}$ in $\mathbb{T}$ is multipliable if and only if the family $(|1-t_\alpha|)_{\alpha\in A}$ is summable in $\mathbb{R}$. The topology of $\ell_1(A,\mathbb{T})$ can be generated by means of one of the following quasinorms:*

$$(t_\alpha) \mapsto ||(t_\alpha)||_1 = \sum_{\alpha\in A} |1-t_\alpha|, \quad (t_\alpha) \mapsto ||(t_\alpha)||'_1 = \sup_{\Delta\in\mathcal{F}(A)} |1 - \prod_{\alpha\in\Delta} t_\alpha|.$$

Moreover, we have $||\cdot||'_1 \le ||\cdot||_1$ and

$$(t_\alpha) \in \ell_1(A,\mathbb{T}), \quad ||(t_\alpha)||'_1 \le \frac{\sqrt{2}}{2} \quad \Longrightarrow \quad ||(t_\alpha)||_1 \le \frac{\pi}{\sqrt{2}} ||(t_\alpha)||'_1. \tag{3.2}$$

Proof. The first part is well known (cf. [9, VIII.16]); it follows also from Lemma 2.2(b). It is clear from (2.2) in Lemma 2.1(a) that $||\cdot||'_1$ really generates the topology of $\ell_1(A,\mathbb{T})$. The inequality $||\cdot||'_1 \le ||\cdot||_1$ and (3.2) show that $||\cdot||_1$ and $||\cdot||'_1$ generate the same topology. The inequality is evident; (3.2) follows from Lemma 2.2(b). □

Proposition 3.3. *Let G be a locally quasi-convex group. Then the topology of $\ell_1(A, G)$ can be generated by the family of quasinorms*

$$(x_\alpha) \mapsto s_V((x_\alpha)) := \sup_{\Delta \in \mathcal{F}(A)} \| \sum_{\alpha \in \Delta} x_\alpha \|_V,$$

where V runs over all quasi-convex neighborhoods of zero in G. More precisely, if $V \in \mathcal{N}_0(G)$ is quasi-convex, then

$$(V)_A = \{(x_\alpha)_{\alpha \in A} \in \ell_1(A, G) : s_V((x_\alpha)) \leq \sqrt{2}\}.$$

Proof. Take into account that if $V \in \mathcal{N}_0(G)$ is quasi-convex, then $V = \{x \in G : ||x||_V \leq \sqrt{2}\}$. □

Proposition 3.4. *Let G be a topological Abelian group and A an infinite set.*

(a) *If G is complete then $\ell_1(A, G)$ is complete.*
(b) *If G is metrizable then $\ell_1(A, G)$ is metrizable.*
(c) *If A is countable and G is metrizable and separable then $\ell_1(A, G)$ is metrizable and separable.*

Proof. Straightforward. □

Let us consider now the notion of absolute summability in groups. A family $(x_\alpha)_{\alpha \in A}$ in a topological Abelian group G is said to be *absolutely summable* if for every $U \in \mathcal{N}_0(G)$ the scalar family $(k_U(x_\alpha))$ is summable in $\mathbb{R}$ (here k_U is the Kaplan functional introduced in Section 2).

A similar definition has been used in [6]. The set of all absolutely summable families with index set A in a topological Abelian group G will be denoted by $\ell_1\{A, G\}$, or $\ell_1\{G\}$ if $A = \mathbb{N}$.

Proposition 3.5. *Let A be an infinite set and G a topological Abelian group. Then*

(a) *$\ell_1\{A, G\}$ is a subgroup of G^A; hence itself is a group with respect to the group operation induced by G^A.*
(b) *The sets having the form*

$$\{V\}_{\varepsilon,A} = \{(x_\alpha)_{\alpha \in A} \in \ell_1\{A, G\} \ : \ \widetilde{k_V}((x_\alpha)) := \sum_{\alpha \in A} k_V(x_\alpha) \leq \varepsilon\},$$

where V runs over $\mathcal{N}_0(G)$ and $\varepsilon \in (0, 1]$, form a fundamental system of neighborhoods of zero for a group topology $\{\tau_G\}_A$ in $\ell_1\{A, G\}$.

Proof. This follows directly from Lemma 2.4, (c) and (d). □

In what follows we shall equip the group $\ell_1\{A, G\}$ with the topology $\{\tau_G\}_A$ that we have introduced in the Proposition 3.5.

Lemma 3.6. *Let G be a topological Abelian group, A an infinite set and H a subgroup of G with the topology induced from G. Then*

(a) $\ell_1(A, H) = \ell_1(A, G) \cap H^A, \quad \ell_1\{A, H\} = \ell_1\{A, G\} \cap H^A.$
(b) *for every $V \in \mathcal{N}_0(G)$ and $\varepsilon \in (0, 1]$*

$$(V \cap H)_A = (V)_A \cap H^A \text{ and } \{V \cap H\}_{\varepsilon,A} = \{V\}_{\varepsilon,A} \cap H^A.$$

(c) *the canonical inclusion maps* $\ell_1(A,H) \hookrightarrow \ell_1(A,G)$ *and* $\ell_1\{A,H\} \hookrightarrow \ell_1\{A,G\}$ *are topological embeddings.*

Proof. (a) and (b) can be directly deduced from the definitions. (c) follows from (b). □

Lemma 3.7. *A family* $(x_\alpha)_{\alpha\in A}$ *in* $\mathbb{R}$ *is absolutely summable if and only if the family* $(|x_\alpha|)_{\alpha\in A}$ *is summable in* $\mathbb{R}$*. The topology of* $\ell_1\{A,\mathbb{R}\}$ *can be generated by the norm*

$$(x_\alpha)_{\alpha\in A} \mapsto \sum_{\alpha\in A} |x_\alpha|.$$

Consequently, $\ell_1\{A,\mathbb{R}\}$ *and* $\ell_1(A,\mathbb{R})$ *coincide algebraically and topologically.*

Proof. This follows directly from Lemma 2.4(f) and Lemma 3.1. □

Lemma 3.8. *Let* $(t_\alpha)_{\alpha\in A}$ *be a family in* $\mathbb{T}$*. Then* $(t_\alpha)_{\alpha\in A} \in \ell_1\{A,\mathbb{T}\}$ *if and only if* $(t_\alpha)_{\alpha\in A}$ *is multipliable in* $\mathbb{T}$*. The topology of* $\ell_1\{A,\mathbb{T}\}$ *can be generated by the quasinorm* $||\cdot||_1$ *of Lemma 3.2. Consequently,* $\ell_1\{A,\mathbb{T}\}$ *and* $\ell_1(A,\mathbb{T})$ *coincide algebraically and topologically.*

Proof. This follows directly from Lemma 2.4(e) and Lemma 3.2. □

Proposition 3.9. *Let* G *be a locally quasi-convex topological Abelian group.*

(a) *A family* (x_α) *in* G *is absolutely summable if and only if for every* $V \in \mathcal{N}_0(G)$

$$r_V((x_\alpha)) := \sum_{\alpha\in A} \|x_\alpha\|_V < \infty.$$

(b) *The topology of* $\ell_1\{A,G\}$ *can be defined by means of the quasinorms* $\{r_V : V \in \mathcal{N}_0(G)\}$*. More precisely, the sets*

$$[V]_A = \{(x_\alpha)_{\alpha\in A} \in \ell_1\{A,G\} : r_V((x_\alpha)) \le \sqrt{2}\},$$

where V *runs over all quasi-convex neighborhoods of zero in* G*, form a fundamental system of neighborhoods of zero for* $\{\tau_G\}_A$*.*

(c) *If* (x_α) *is an absolutely summable family in* G *then it is presummable, i. e.* $\ell_1\{A,G\} \subset \ell_1(A,G)$*. The corresponding inclusion map is continuous.*

Proof. (a) This follows at once from Lemma 2.5.

(b) Again from Lemma 2.5 it follows easily that for any quasi-convex $V \in \mathcal{N}_0(G)$,

$$\{V\}_{\frac{\sqrt{2}}{\pi},A} \subset [V]_A \subset \{V\}_{1,A}.$$

(c) Fix $V \in \mathcal{N}_0(G)$. Since $(x_\alpha)_{\alpha\in A}$ is absolutely summable, the family $(k_V(x_\alpha))_{\alpha\in A}$ is summable in $\mathbb{R}$. By Lemma 2.5, the family $(\|x_\alpha\|_V)_{\alpha\in A}$ also is summable. Hence there is a finite subset $\Delta_0 \subset A$ such that for any fixed $\Delta \in \mathcal{F}(A)$ with $\Delta\cap\Delta_0 = \emptyset$ we have

$$\|\sum_{\alpha\in\Delta} x_\alpha\|_V \le \sum_{\alpha\in\Delta} \|x_\alpha\|_V \le \sqrt{2}.$$

Suppose now that V is quasi-convex. Then the above inequality implies $\sum_{\alpha\in\Delta} x_\alpha \in V$. Since G is locally quasi-convex, we obtain that (x_α) is presummable in G. The continuity of the inclusion map follows from the inclusion $[V]_A \subset (V)_A$, which is valid for any quasi-convex $V \in \mathcal{N}_0(G)$.

□

Remark 3.10. The assertion in (c) will play an important role in what follows. Its algebraic part was obtained previously in [6, p. 277]. This statement justifies the name "absolutely summable family" in the case of locally quasi-convex groups. The local quasi-convexity assumption is essential for its validity, since for a non-locally quasi-convex group absolute summability may not imply presummability (cf. [18, p. 315]).

Proposition 3.11. *Let G be a locally quasi-convex topological Abelian group. Then Proposition 3.4 remains valid also for $\ell_1\{A, G\}$.*

Proof. Straightforward. □

4. GP-nuclear groups

In the previous section, for any given topological Abelian group G and any infinite set of indices A, we have introduced the new topological Abelian groups $\ell_1(A, G)$ and $\ell_1\{A, G\}$ of presummable and absolutely summable families of elements in G. Using these groups, now we can formulate the following definition:

Definition 4.1. A topological Abelian group G is said to be *Grothendieck-Pietsch nuclear,* or simply *GP-nuclear,* if $\ell_1(G) = \ell_1\{G\}$ algebraically and $(\tau_G)_{\mathbb{N}} = \{\tau_G\}_{\mathbb{N}}$.

In what follows we shall deal mainly with locally quasi-convex GP-nuclear groups. Before giving further comments about this notion let us first demonstrate that in the definition, instead of index set $\mathbb{N}$ we could take an arbitrary infinite set A.

Proposition 4.2. *Let G be a locally quasi-convex topological Abelian group. The following are equivalent:*

(i) *G is GP-nuclear.*
(ii) *For any infinite set of indices A, $\ell_1(A, G) = \ell_1\{A, G\}$ and $\{\tau_G\}_A = (\tau_G)_A$.*
(iii) *There is an infinite set of indices A such that $\ell_1(A, G) = \ell_1\{A, G\}$ and $\{\tau_G\}_A = (\tau_G)_A$.*

Proof. (i)⇒(ii): We follow the scheme in [19, p. 182]. Fix A. Because of Proposition 3.9(c), it is needed only to show that $\ell_1(A, G) \subset \ell_1\{A, G\}$ and $\{\tau_G\}_A \subset (\tau_G)_A$. Suppose that the first inclusion is not true. Then there exists a presummable family $(x_\alpha)_{\alpha \in A}$ which is not absolutely summable. Hence, for some $V \in \mathcal{N}_0(G)$ the scalar family $(k_V(x_\alpha))_{\alpha \in A}$ is not summable. This implies that there exists a countable subset $A_0 \subset A$ such that $(k_V(x_\alpha))_{\alpha \in A_0}$ is not summable. But $(x_\alpha)_{\alpha \in A}$ was presummable, hence $(x_\alpha)_{\alpha \in A_0}$ is also presummable. Since G is GP-nuclear and A_0 is countable, this implies that $(x_\alpha)_{\alpha \in A_0}$ is absolutely summable and, in particular, that $(k_V(x_\alpha))_{\alpha \in A_0}$ is summable, a contradiction.

Let us show now that $\{\tau_G\}_A \subset (\tau_G)_A$. Fix $U \in \mathcal{N}_0(G)$; we shall find $V \in \mathcal{N}_0(G)$ such that $(V)_A \subset [U]_A$. Since G is GP-nuclear, $\{\tau_G\}_{\mathbb{N}} \subset (\tau_G)_{\mathbb{N}}$. Hence, there exists $V \in \mathcal{N}_0(G)$ such that $(V)_{\mathbb{N}} \subset [U]_{\mathbb{N}}$. Now it is easy to see that we have also $(V)_A \subset [U]_A$.

(iii)⇒(i): We can suppose that $\mathbb{N} \subset A$. Let us show that $\ell_1(\mathbb{N}, G) \subset \ell_1\{\mathbb{N}, G\}$. Take $(x_n) \in \ell_1(\mathbb{N}, G)$. Consider the family $(y_\alpha)_{\alpha \in A}$ defined as follows: $y_\alpha = x_\alpha$ if $\alpha \in \mathbb{N}$, $y_\alpha = 0$ otherwise. The fact that (x_n) is presummable implies easily that (y_α) is also presummable. By our assumption $(y_\alpha) \in \ell_1\{A, G\}$. This again implies that $(x_n) \in \ell_1\{\mathbb{N}, G\}$.

Let us show now that $(\tau_G)_{\mathbb{N}} \subset \{\tau_G\}_{\mathbb{N}}$. Fix any $V \in \mathcal{N}_0(G)$. By our supposition we have $(\tau_G)_A \subset \{\tau_G\}_A$. Hence there exists $U \in \mathcal{N}_0(G)$ such that $(U)_A \subset [V]_A$. It is easy to check that this implies $(U)_{\mathbb{N}} \subset [V]_{\mathbb{N}}$. □

Note that at the level of the definition we can say only that the groups $\mathbb{R}$ and $\mathbb{T}$ are GP-nuclear (see Lemma 3.7 and Lemma 3.8). To provide more examples of GP-nuclear groups and also to show that any nuclear group is GP-nuclear, some preparation is needed.

Proposition 4.3. *Let G be a topological Abelian group and H a topological subgroup of G.*

(a) *If $\ell_1(G) = \ell_1\{G\}$ then $\ell_1(H) = \ell_1\{H\}$.*
(b) *If G is GP-nuclear, then H is GP-nuclear.*

Proof. (a) follows from Lemma 3.6(a). (b) follows from (a) and Lemma 3.6(c). □

Proposition 4.4. *Let G be a locally quasi-convex metrizable topological Abelian group and H be a dense subgroup of G. If in the topological group H any presummable sequence is absolutely summable, then the same is true for G.*

Proof. It is possible to construct a fundamental sequence of neighborhoods of zero in G, (V_n), such that V_n is symmetric for every $n \in \mathbb{N}$ and

$$y_n \in V_n \quad \forall n \in \mathbb{N} \Rightarrow (y_n) \in \ell_1\{G\}.$$

Let (x_n) be a presummable sequence in G. Let (z_n) be a sequence in H such that $z_n - x_n \in V_n$ for every $n \in \mathbb{N}$; hence, the sequence $(z_n - x_n)$ is absolutely summable and by Prop. 3.9(c), it is also presummable in G. Therefore, $(z_n) = (z_n - x_n) + (x_n)$ is presummable in G and, by Lemma 3.6(a), also in H. By hypothesis, (z_n) is absolutely summable in H and, again by Lemma 3.6(a), also in G. Therefore $(x_n) = (x_n - z_n) + (z_n)$ is absolutely summable in G. □

Proposition 4.5. *Let G be a topological Abelian group, G_i, $i \in I$ a family of locally quasi-convex topological Abelian groups, $u_i : G \to G_i$, $i \in I$ group homomorphisms. Suppose that the topology in G is coarsest among the topologies in G which make u_i continuous for every $i \in I$.*

(a) *If $\ell_1(G_i) = \ell_1\{G_i\}$ for every $i \in I$, then $\ell_1(G) = \ell_1\{G\}$.*
(b) *If G_i, $i \in I$ are GP-nuclear then G is GP-nuclear.*

Proof. (a) It is easy to show that G is locally quasi-convex, so by Proposition 3.9(c) only the inclusion $\ell_1(G) \subset \ell_1\{G\}$ remains to be proved. The following family can be taken as a basis of neighborhoods of zero for the topology of G :

$$\{\bigcap_{j=1}^{n} u_{i_j}^{-1}(U_j) : U_j \in \mathcal{N}_0(G_{i_j}),\ j = 1, \dots, n,\ n \in \mathbb{N}\}$$

Let $(x_m) \in \ell_1(G)$ and $U = \bigcap_{j=1}^{n} u_{i_j}^{-1}(U_j)$ as above. Let us show that $\sum_{m=1}^{\infty} k_U(x_m) < \infty$. Observe that

$$k_U(x) = \max_{j \in \{1,\dots,n\}} k_{U_j}(u_{i_j}(x)) \quad \forall x \in G$$

and that, since u_i is continuous, the sequence $(u_i(x_m))_{m\in\mathbb{N}}$ is presummable and, hence, absolutely summable in G_i for every $i \in I$. Therefore

$$\sum_{m=1}^{\infty} k_U(x_m) = \sum_{m=1}^{\infty} \max_{j\in\{1,\dots,n\}} k_{U_j}(u_{i_j}(x_m))$$
$$\leq \sum_{m=1}^{\infty}\sum_{j=1}^{n} k_{U_j}(u_{i_j}(x_m)) = \sum_{j=1}^{n}\sum_{m=1}^{\infty} k_{U_j}(u_{i_j}(x_m)) < \infty.$$

(b) By (a) above and Proposition 3.9(c), it remains only to prove that the identity $\ell_1(G) \to \ell_1\{G\}$ is continuous. Let $U = \bigcap_{j=1}^n u_{i_j}^{-1}(U_j)$ as above, and $\varepsilon \in (0, 1]$. We shall find $V \in \mathcal{N}_0(G)$ such that $(V)_{\mathbb{N}} \subset \{U\}_{\varepsilon,\mathbb{N}}$. Since for every $i \in I$, the identity map $\ell_1(G_i) \to \ell_1\{G_i\}$ is continuous, for every $j \in \{1, \dots, n\}$ there exists $V_j \in \mathcal{N}_0(G_{i_j})$ such that $(V_j)_{\mathbb{N}} \subset \{U_j\}_{\frac{\varepsilon}{n},\mathbb{N}}$. Let $V = \bigcap_{j=1}^n u_{i_j}^{-1}(V_j)$. For every $(x_m)_{m\in\mathbb{N}} \in (V)_{\mathbb{N}}$ and every $j \in \{1, \dots, n\}$ we have $(u_{i_j}(x_m))_{m\in\mathbb{N}} \in (V_j)_{\mathbb{N}} \subset \{U_j\}_{\frac{\varepsilon}{n},\mathbb{N}}$. Therefore

$$\sum_{m=1}^{\infty} k_U(x_m) = \sum_{m=1}^{\infty} \max_{j\in\{1,\dots,n\}} k_{U_j}(u_{i_j}(x_m))$$
$$\leq \sum_{j=1}^{n}\sum_{m=1}^{\infty} k_{U_j}(u_{i_j}(x_m)) \leq n\frac{\varepsilon}{n} = \varepsilon,$$

hence $(x_m) \in \{U\}_{\varepsilon,\mathbb{N}}$ and $(V)_{\mathbb{N}} \subset \{U\}_{\varepsilon,\mathbb{N}}$.

□

Corollary 4.6. (a) *The Cartesian product of any family of locally quasi-convex GP-nuclear groups is GP-nuclear.*
(b) *Any topological Abelian group, endowed with its Bohr topology, is GP-nuclear.*

Proof. This follows directly from Proposition 4.5(b) and from the definition of the product topology (in the case of (a)) and the definition of the Bohr topology (in the case of (b)). Recall that the Bohr topology of G is by definition the coarsest topology on G which makes all characters $\varphi \in G^\wedge$ continuous. □

Remark 4.7. Using Corollary 4.6(a), Proposition 4.3(b) and the structure theorems for locally compact Abelian groups, it is easy to show that these groups are GP-nuclear. However, this follows also from Theorem 4.10 below.

Next we shall prove that, under the assumptions of metrizability and local quasi-convexity on the group G, the set-theoretical identity between the groups of presummable and absolutely summable families implies the topological equivalence.

Theorem 4.8. *Let G be a metrizable locally quasi-convex topological Abelian group. Then G is GP-nuclear if and only if $\ell_1(G) = \ell_1\{G\}$ algebraically.*

Proof. Step 1. Suppose first that G is separable and complete. By Proposition 3.9(c) the inclusion map $i : \ell_1\{G\} \to \ell_1(G)$ is continuous and by our supposition it is also surjective. According to Propositions 3.4 and 3.11, $\ell_1(G)$ and $\ell_1\{G\}$ are also metrizable, separable and complete. From the open homomorphism theorem (see e. g. [9, IX.116, ex. 28]) it follows that i is open, hence i is a homeomorphism.

Step 2. If G is separable but not complete, consider the completion $\widetilde{G}$ of G. $\widetilde{G}$ is complete, metrizable, separable, locally quasi-convex ([8]). Since $\ell_1(G) = \ell_1\{G\}$, by Proposition 4.4 we obtain that $\ell_1(\widetilde{G}) = \ell_1\{\widetilde{G}\}$. According to Step 1, we can conclude that $\widetilde{G}$ is GP-nuclear. Hence, by Proposition 4.3(b), its subgroup G is also GP-nuclear.

Step 3. Now we consider the general case. Suppose that there exist $V \in \mathcal{N}_0(G)$ and $\varepsilon \in (0,1]$ such that for every $W \in \mathcal{N}_0(G)$, $(W)_{\mathbb{N}} \not\subset \{V\}_{\varepsilon,\mathbb{N}}$. Let $(W_n)_{n\in\mathbb{N}}$ be a fundamental sequence of neighborhoods of zero in G. For every $n \in \mathbb{N}$ we can find a sequence $(x_{n,k})_{k\in\mathbb{N}} \in (W_n)$ such that $(x_{n,k})_{k\in\mathbb{N}} \notin \{V\}_{\varepsilon,\mathbb{N}}$. Consider the subgroup H generated by $\{x_{n,k} : n \in \mathbb{N}, k \in \mathbb{N}\}$. H is countable (hence separable), metrizable and by Prop. 4.3(a), we have $\ell_1(H) = \ell_1\{H\}$. According to Step 2, H is GP-nuclear, hence $(\tau_H)_{\mathbb{N}} = \{\tau_H\}_{\mathbb{N}}$, but $(W_n \cap H)_{\mathbb{N}} \not\subset \{V \cap H\}_{\varepsilon,\mathbb{N}}$ for every $n \in \mathbb{N}$ (see Lemma 3.6(b)), a contradiction. □

Remark 4.9. The assumption of metrizability is essential for the validity of Theorem 4.8. See Remark 5.9 below.

Now we can formulate our final result:

Theorem 4.10. *Let G be a nuclear group. Then G is GP-nuclear.*

Proof. Step 1. G is locally quasi-convex ([2, 8.5]) and $\ell_1(G) = \ell_1\{G\}$ algebraically ([6, Th. 3]).

Step 2. If G is a metrizable nuclear group, then GP-nuclearity of G follows from Step 1 and Theorem 4.8.

Step 3. Every nuclear group can be embedded into a Cartesian product of metrizable nuclear groups ([1, 21.3]). Now GP-nuclearity of G follows from Step 2, Corollary 4.6 and Proposition 4.3(b). □

As we have noted in the Introduction, the validity of converse assertion remains open. We did not succeed to construct any counterexample. By this reason, we think that any locally quasi-convex GP-nuclear group should be nuclear. That is why we are not giving here other properties of GP-nuclear groups. Let us mention only that the topological direct sum of countably many GP-nuclear groups is GP-nuclear.

In the next section we shall show that the additive topological Abelian group of a locally convex space E is GP-nuclear if and only if E is a nuclear locally convex space.

5. The case of topological vector spaces

In this section we shall check that the notions which we have considered before in the framework of groups have their ordinary meaning in the case of additive subgroups of topological vector spaces.

Let E be a (real) topological vector space. E^* denotes its topological dual space, which consists of all continuous linear functionals $x^* : E \to \mathbb{R}$. Again, no topology on E^* is fixed.

For a nonempty subset $V \subset E$, V° will denote the ordinary (absolute) polar of V, i. e. $V^\circ = \{x^* \in E^* : |x^*(x)| \leq 1 \; \forall x \in V\}$. Also, for a nonempty subset $B \subset E^*$ we shall denote by ${}^\circ B$ its inverse polar, i. e. ${}^\circ B = \{x \in E : |x^*(x)| \leq 1 \; \forall x^* \in B\}$.

If V is a nonempty subset of E, then p_V denotes its Minkowski functional, i. e. $p_V(x) = \inf\{\lambda > 0 : x \in \lambda V\}$. We agree that $\inf \emptyset = \infty$.

Lemma 5.1. *If $V \in \mathcal{N}_0(E)$ is absolutely convex, then*

$$p_V(x) = \sup_{x^* \in V^\circ} |x^*(x)| \quad \forall x \in E.$$

Proof. This assertion is a standard corollary of the Hahn-Banach theorem. □

Lemma 5.2. *If V is a radial subset of E (i. e. $rV \subset V \; \forall r \in [0, 1]$), then*

$$k_V(x) \leq p_V(x) \quad \forall x \in E \tag{5.1}$$

and

$$p_V(x) \leq 2k_V(x) \quad \forall x \in V. \tag{5.2}$$

Proof. Suppose first that $x \notin V$. Then, since V is radial, $p_V(x) \geq 1$ and $k_V(x) = 1 \leq p_V(x)$.

Suppose now that $x \in V$. If $k_V(x) = 0$ then $nx \in V$ for every $n \in \mathbb{N}$. Hence, $p_V(x) = 0$ and in this case (5.1) and (5.2) are proved. If $k_V(x) = \frac{1}{n+1}$ for some $n \in \mathbb{N}$, then $nx \in V$ and $(n+1)x \notin V$. The first condition implies (5.2) and (5.1) follows from the second one, since V is radial. □

Since E is also an topological Abelian group, it has a topological dual group $E^\wedge$. The mapping $x^* \to \exp(2\pi i x^*)$ establishes an algebraic isomorphism between the additive group E^* and the multiplicative group $E^\wedge$.

Lemma 5.3. *Let U be a nonempty radial subset of E. Then*

(a) $U^\triangleright = \{\exp(\frac{\pi}{2} i x^*) : x^* \in U^\circ\}$, *and the equality*

$$\|x\|_U = \sup_{x^* \in U^\circ} 2\sin(\frac{\pi}{4}|x^*(x)|) \quad \forall x \in E \tag{5.3}$$

holds.

(b) $\|x\|_U \leq \frac{\pi}{2} p_U(x) \quad \forall x \in E$; *more precisely,* $\|x\|_U \leq 2\sin(\frac{\pi}{4} p_U(x)) \quad \forall x \in U$, *and if U is an absolutely convex neighborhood of zero, then $p_U(x) \leq \frac{1}{\sqrt{2}}\|x\|_U$ for every $x \in U$.*

(c) *If $U \in \mathcal{N}_0(E)$, then U is quasi-convex in the additive group of E if and only if U is closed and absolutely convex.*

Proof. (a) Let $B = \{\exp(\frac{\pi}{2} i x^*\} : x^* \in U^\circ\}$. The inclusion $B \subset U^\triangleright$ is clear and valid for any U. Let now $\varphi \in E^\wedge$. Then $\varphi = \exp(2\pi i y^*)$ for some $y^* \in E^*$. Suppose $\varphi \in U^\triangleright$ and fix a nonzero $x \in U$. Then $|1 - \varphi(x)| = 2\sin|\pi y^*(x)| \leq \sqrt{2}$. Since U is radial we can write also, for any fixed $r \in [0, 1]$, that $|1 - \varphi(rx)| = 2\sin|\pi y^*(rx)| \leq \sqrt{2}$. This evidently implies that $|y^*(x)| \leq \frac{1}{4}$. Hence $x^* = 4y^* \in U^\circ$ and $\varphi \in B$.

(b) The first inequality follows from (5.3), since $|x^*(x)| \le p_U(x)\ \ \forall x \in E\ \ \forall x^* \in U^\circ$.
Using the same reasoning and the fact that $\sin\xi$ is monotone in $[0, \frac{\pi}{4}]$ we obtain the second inequality.
The third one is consequence of the equality (5.3), the inequality $\sin\xi \ge \frac{2\sqrt{2}}{\pi}\xi$ (valid for $\xi \in [0, \frac{\pi}{4}]$) and Lemma 5.1.

(c) This follows from (5.3) in (a) and Lemma 5.1, taking into account that U is closed and absolutely convex if and only if $U = \{x \in E : p_U(x) \le 1\}$, and quasi-convex if and only if $U = \{x \in E : \|x\|_U \le \sqrt{2}\}$.

□

Using this Lemma, the following result can be deduced, which is an important justification for the notion of locally quasi-convex group (see [2, 2.4]):

Lemma 5.4. *The additive group of a topological vector space E is locally quasi-convex if and only if E is locally convex.*

Let, as in previous sections, A be an infinite set of indices. If E is a topological vector space then the group of all presummable families $\ell_1(A, E)$ is also a vector space. The space $\ell_1(A, E)$ where E is a locally convex space has been studied in detail in [17], where it is denoted by $l^1_A(E)$ and it is equipped with the ε-topology, generated by the seminorms

$$(x_\alpha)_{\alpha\in A} \mapsto \varepsilon_U((x_\alpha)) = \sup_{x^*\in U^\circ} \sum_{\alpha\in A} |x^*(x_\alpha)|,$$

where U runs over all absolutely convex neighborhoods of zero in E. In Section 3 we have introduced in $\ell_1(A, E)$ the topology $(\tau_E)_A$.

Proposition 5.5. *If E is a locally convex space then for an arbitrary infinite set of indices A the $\varepsilon-$topology in $\ell_1(A, E)$ coincides with $(\tau_E)_A$.*

Proof. By Lemma 5.4, E is locally quasi-convex. By Proposition 3.3 and Lemma 5.3(c), $(\tau_E)_A$ is generated by the quasinorms s_V, where V runs over all closed and absolutely convex $V \in \mathcal{N}_0(E)$.

Fix a closed and absolutely convex $V \in \mathcal{N}_0(E)$. Using (3.1) in Lemma 3.1, Lemma 5.3(b) and Lemma 5.1 we obtain

$$s_V((x_\alpha)) \le \frac{\pi}{2}\varepsilon_V((x_\alpha)) \quad \forall (x_\alpha) \in \ell_1(A, E)$$

and

$$\varepsilon_V((x_\alpha)) \le \sqrt{2}s_V((x_\alpha)) \quad \forall (x_\alpha) \in (V)_A.$$

This implies the assertion. □

The following assertion shows that in the case of topological vector spaces our notion of absolute summability coincides with the similar notion used for this case for instance in [13], [18] and, for locally convex spaces, in [17].

Proposition 5.6. *Let E be a topological vector space, $(x_\alpha)_{\alpha\in A}$ a family of elements of E. Then (x_α) is absolutely summable if and only if for every $V \in \mathcal{N}_0(E)$, the family $(p_V(x_\alpha))_{\alpha\in A}$ is summable in $\mathbb{R}$.*

Proof. This follows from Lemma 5.2. □

Let E be a locally convex space. Then the group $\ell_1\{A, E\}$ is also a vector space. This space is denoted in [17] by $l^1_A\{E\}$ and it is equipped with the π-topology, generated by the seminorms

$$(x_\alpha)_{\alpha\in A} \mapsto \pi_U((x_\alpha)) = \sum_{\alpha\in A} p_U(x_\alpha),$$

where U runs over all absolutely convex neighborhoods of zero in E.

Proposition 5.7. *If E is a locally convex space then for an arbitrary infinite set of indices A the π-topology in $\ell_1\{A, E\}$ coincides with $\{\tau_E\}_A$.*

Proof. Fix a closed and absolutely convex $V \in \mathcal{N}_0(E)$ and $(x_\alpha) \in \ell_1\{A, E\}$. Using Lemma 5.2, we can conclude that $\widetilde{k_V}((x_\alpha)) \leq \pi_V((x_\alpha))$ and

$$\widetilde{k_V}((x_\alpha)) < 1 \Rightarrow \pi_V((x_\alpha)) \leq 2\widetilde{k_V}((x_\alpha)).$$

These inequalities imply the assertion. □

Taking into account the above Propositions and the Grothendieck-Pietsch theorem mentioned in the Introduction, we can state:

Theorem 5.8. *The additive topological Abelian group of a locally convex space E is GP-nuclear if and only if E is a nuclear space.*

Remark 5.9. Let A be an uncountable set of indices and $E = \mathbb{R}^A_0$ be the locally convex direct sum of A copies of $\mathbb{R}$. It is easy to observe that $\ell_1(E) = \ell_1\{E\}$ algebraically. However, E is not nuclear ([17, (4.3.4), p. 78]), hence by Theorem 5.8 it is not GP-nuclear. This shows that Theorem 4.8 is not true without the assumption of metrizability.

In the case of metrizable spaces we have also the following

Proposition 5.10. *The additive topological Abelian group of a metrizable topological vector space E is GP-nuclear if and only if E is a nuclear locally convex space.*

Proof. Since E is GP-nuclear, in particular, we have $\ell_1\{E\} \subset \ell_1(E)$. This, according to [18, p. 315], implies that E is locally convex. It remains to apply Theorem 5.8. □

Remark 5.11. There exists also a notion of metrizable nuclear topological vector space in which local convexity is not required ([18, p. 296]). The above Proposition shows that the notion of GP-nuclear space is more restrictive.

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